The Process of Integration by Parts[1]

William D. Emerson

Department of Mathematical and Computer Sciences

Metropolitan State College of Denver

The two most fundamental techniques of integration are the method of substitution and integration by parts. The method of substitution is an integral expression of the chain rule, while the method of integration by parts corresponds to the product rule for differentiation. In this short note we derive the formula for integration by parts and look at several examples of how this technique may be applied to integrals.

Derivation of the formula. The product rule for differentiation is

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

If we integrate with respect to x we obtain

$$\int (f(x)g(x))' \, dx = \int f'(x)g(x) \, dx + \int f(x)g'(x) \, dx.$$

Since $\int (f(x)g(x))' \, dx = f(x)g(x)$ (integrating after differentiating takes us back to where we started) we have

$$f(x)g(x) = \int f'(x)g(x) \, dx + \int f(x)g'(x) \, dx.$$

Solving for $\int f'(x)g(x) \, dx$ gives the following integration by parts formula for indefinite integrals:

$$\boxed{\int f'(x)g(x) \, dx = f(x)g(x) - \int f(x)g'(x) \, dx.}$$

Replacing the indefinite integrals above with definite integrals yields the integration by parts formula for definite integrals:

$$\boxed{\int_a^b f'(x)g(x) \, dx = f(x)g(x)\Big|_a^b - \int_a^b f(x)g'(x) \, dx.}$$

The Process of Integration by Parts. We have found that thinking of integration by parts as a process, rather than as a formula, allows us to apply

[1] I want to thank Dr. David S. Gilliam, Texas Tech University, for first showing me this "alternate" approach to integration by parts.

the technique more quickly and easily. This process consists of two steps, an integration followed by a differentiation. Beginning with the integration problem

$$\int f'(x)g(x)\ dx$$

we integrate $f'(x)$ to obtain $f(x)$. This first step allows us to write

$$\int f'(x)g(x)\ dx = f(x)g(x) + \cdots.$$

For the last term in the formula,

$$\int f(x)g'(x)\ dx$$

we need to know $g'(x)$. We obtain this easily by differentiating $g(x)$. For our second step we may now write

$$\int f'(x)g(x)\ dx = f(x)g(x) - \int f(x)g'(x)\ dx.$$

If we want to evaluate our given integral we still need to evaluate the last integral

$$\int f(x)g'(x).$$

Our usual goal in using the integration by parts process is that this will be an integral that is more easily handled than our original integral.

Example 1. Use integration by parts to evaluate

$$\int x\cos x\ dx.$$

Solution. We have a product, $x\cos x$, and we need to choose one of these functions to integrate. We choose $\cos x$ and note that its integral is $\sin x$. This allows us to write down the first step:

$$\int x\cos x\ dx = x\sin x - \ldots$$

In the first step we integrated $\cos x$ so, in the second step, we will differentiate the other function, x. The derivative of x is 1 so we have

$$\begin{aligned}\int x\cos x\ dx &= x\sin x - \int 1\cdot\sin x\ dx\\ &= x\sin x - \int \sin x\ dx.\end{aligned}$$

At this point we have completed the integration by parts process. However, we are not finished. We must still evaluate the last integral on the right. Since $\int \sin x\ dx = -\cos x$ we may write

$$\begin{aligned}\int x\cos x\ dx &= x\sin x - (-\cos x) + C\\ &= x\sin x + \cos x + C.\end{aligned}$$

Note. If we had chosen to integrate x as the first step in this example we would have obtained

$$\begin{aligned}\int x\cos x dx &= \frac{x^2}{2}\cos x - \cdots \\ &= \frac{x^2}{2}\cos x - \frac{1}{2}\int x^2 \cdot -\sin x \, dx \\ &= \frac{x^2}{2}\cos x + \frac{1}{2}\int x^2 \sin x \, dx.\end{aligned}$$

Although the process of integration by parts has been carried out correctly, the integral

$$\int x^2 \sin x \, dx$$

is more difficult to evaluate than our original problem. This example shows that it is important to make the right choice of function to integrate in the first step. (Choosing the right function becomes easier with practice.)

Example 2. Use integration by parts to evaluate

$$\int \ln x \, dx$$

Solution. The integration by parts process is applied to products and we don't have a product in this integral. However we can make this into a product by writing

$$\int \ln x \, dx = \int 1 \cdot \ln x \, dx.$$

Now we must choose to integrate either $\ln x$ or 1. If we could integrate $\ln x$ easily we could have evaluated our original integral easily. Thus, we integrate 1 to obtain x and write

$$\int 1 \cdot \ln x \, dx = x \cdot \ln x - \cdots$$

The derivative of $\ln x$ is $\frac{1}{x}$ so we have

$$\begin{aligned}\int 1 \cdot \ln x \, dx &= x\ln x - \int x\frac{1}{x} \, dx \\ &= x\ln x - \int 1 \, dx \\ &= x\ln x - x + C\end{aligned} \tag{1}$$

Example 3. Use integration by parts to evaluate

$$\int x^2 \sin x \, dx.$$

Solution. We'll start by integrating $\sin x$:

$$\int x^2 \sin x \; dx = x^2 \cdot -\cos x - ...$$

Now we can take the derivative of x^2 to obtain

$$\begin{aligned} \int x^2 \sin x \; dx &= x^2 \cdot -\cos x - \int 2x \cdot -cosx \; dx \\ &= -x^2 \cos x + 2 \int x \cos x \; dx. \end{aligned}$$

We've succeeded in lowering the power of x from 2 to 1. We can apply integration by parts to $\int x \cos x \; dx$ by integrating $\cos x$ and differentiating x. We obtain:

$$\begin{aligned} \int x^2 \sin x \; dx &= -x^2 \cos x + 2 \int x \cos x \; dx. \\ &= -x^2 \cos x + 2 \left(x \sin x - \int 1 \cdot \sin x \; dx \right) \end{aligned}$$

Finally, the integral of $\sin x$ is $-\cos x$ so we get

$$\begin{aligned} \int x^2 \sin x \; dx &= -x^2 \cos x + 2 \left(x \sin x - (-\cos x) \right) + C \\ &= -x^2 \cos x + 2x \sin x + 2 \cos x + C \end{aligned} \qquad (2)$$

Example 4. Prove the reduction formula

$$\int \cos^n x \; dx = \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n} \int \cos^{n-2} x \; dx$$

Solution. We need a product to use integration by parts so we write

$$\int \cos^n x \; dx = \int \cos x \cos^{n-1} x \; dx.$$

Now we apply integration by parts by integrating $\cos x$ and differentiating $\cos^{n-1} x$ to get

$$\begin{aligned} \int \cos^n x \; dx &= \int \cos x \cos^{n-1} x \; dx \\ &= \sin x \cos^{n-1} x - \int \sin x \cdot (n-1) \cos^{n-2} x (-\sin x) \; dx \\ &= \sin x \cos^{n-1} x + (n-1) \int \sin^2 x \cos^{n-2} x \; dx \end{aligned}$$

Since $\sin^2 x = 1 - \cos^2 x$ we can write this last integral in terms of $\cos x$ only.

$$\int \cos^n x \; dx = \sin x \cos^{n-1} x + (n-1) \int \sin^2 x \cos^{n-2} x \; dx$$

$$
\begin{aligned}
&= \sin x \cos^{n-1} x + (n-1) \int (1-\cos^2 x) \cos^{n-2} x \; dx \\
&= \sin x \cos^{n-1} x + (n-1) \left(\int \cos^{n-2} x \; dx - \int \cos^n x \; dx \right) \\
&= \sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x \; dx - (n-1) \int \cos^n x \; dx
\end{aligned}
$$

Now we can bring the last integral on the right back to the left hand side to obtain

$$\int \cos^n x \; dx + (n-1) \int \cos^n x \; dx = \sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x \; dx$$

or

$$n \int \cos^n x \; dx = \sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x \; dx$$

Dividing by n gives our result

$$\int \cos^n x \; dx = \frac{1}{n} \sin x \cos^{n-1} x + \frac{(n-1)}{n} \int \cos^{n-2} x \; dx$$

Example 5. Use the reduction formula from the previous example to evaluate $\int \cos^6 x \; dx$.

Solution. The reduction formula states

$$\int \cos^n x \; dx = \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n} \int \cos^{n-2} x \; dx$$

so if we let $n = 6$ we have

$$\int \cos^6 x \; dx = \frac{1}{6} \sin x \cos^5 x + \frac{5}{6} \int \cos^4 x \; dx.$$

Applying the reduction formula again, with $n = 4$ yields

$$\int \cos^6 x \; dx = \frac{1}{6} \sin x \cos^5 x + \frac{5}{6} \left(\frac{1}{4} \sin x \cos^3 x + \frac{3}{4} \int \cos^2 x \; dx \right).$$

We still need to evaluate the last integral. Since

$$\int \cos^2 x \; dx = \int \frac{1}{2} + \frac{\cos 2x}{2} \; dx = \frac{x}{2} + \frac{\sin 2x}{2},$$

we have

$$
\begin{aligned}
\int \cos^6 x \; dx &= \frac{1}{6} \sin x \cos^5 x + \frac{5}{6} \left(\frac{1}{4} \sin x \cos^3 x + \frac{3}{4} \left(\frac{x}{2} + \frac{\sin 2x}{4} \right) \right) \\
&= \frac{1}{6} \sin x \cos^5 x + \frac{5}{24} \sin x \cos^3 x + \frac{15}{48} \sin x \cos x + \frac{15}{48} x.
\end{aligned}
$$