What is a real number?

In this talk we will read together the 1901 translation of an 1872 paper by Richard Dedekind where he very eloquently describes a very interesting and easy-to-understand method to create the set of real numbers from the rational numbers.

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Essays on the Theory of Numbers
I. Continuity and Irrational Numbers

By Richard Dedekind (Oct. 6, 1831 – February 12, 1916)
Translated by Wooster Woodruff Beman, of U. of Michigan.

My website:
http://rowdy.mscd.edu/~loatsj/

You can download the attached paper at this site:
http://www.gutenberg.org/etext/21016
Note you'll need only pages 1 – 18 of that .pdf file.

The following narrative is my notes from reading the article as I prepared for this talk. As such, it is fairly telegraphic, but if you have the article at hand, it should make sense. You can down load it at this site: http://www.gutenberg.org/etext/21016 Note you'll need only pages 1 – 18 of that .pdf file.

We are following Dedekind explicitly when the quotes appear.

p.2
"We assume the arithmetic of the rational numbers."

- Whole numbers
- Natural numbers
- Counting numbers
- Integers
- Rational numbers fractions
- Real numbers
  - Constructible numbers - Greeks
  - Algebraic numbers – Cube root of two, square root of π
  - Transcendental numbers,
    - e,
    - π (Irrat: Lambert 1768) (Transcend Lindeman 1882)
    - open questions on π^e or e^π

"The system of rational numbers forms a one-dimensional linear arrangement "extending to infinity to opposite" directions."

> I think geometrically here, but Dedekind cautions that we are going to do everything via arithmetic.

p3.
We write a = b to mean that a & b represent the same rational numbers.

If a & b are different exactly when a-b is positive or negative. If a - b is positive, we write a > b and b < a.

Three principles:

I. "If a > b and b > c, then a > c. In this case we say b is **between** a and c.

II. "If a and c are different rational numbers, then there are infinitely many rational numbers between them.

III. "If a is any rational number, then all numbers of the system R fall into two classes, A_1 and A_2, each of which contains infinitely many rational numbers; the first class A_1 comprises all numbers a_1 that are less than a, the second class comprises all numbers a_2 that are greater than a." What to do with a ? Dedekind lets it be a member of either class – being the greatest of A_1 if it is in A_1 and being the least member of A_2 if it is in A_2. No matter where the number a is put this is true: every member of A_1 is less than every member of A_2.

Part II of Dedekind's article:

He lectures us on the geometric version of the above facts. Let's read it.

p. 4

Part III of Dedekind's article:

"Of the greatest importance, however, is the fact that in the straight line L there are infinitely many points which correspond to no rational number." Then he explains how the Greeks had constructed incommensurable lengths – e.g. the side and the diagonal of a square."

"If now, as is our desire, we try to follow up arithmetically all phenomena in the straight line, the domain of rational numbers is insufficient and it becomes absolutely necessary that the instrument R constructed by the creation of the rational numbers be essentially improved by the creation of new numbers such that the domain of (real) numbers shall gain the same <u>completeness</u>, or as we may say at once the same <u>continuity</u>, as the straight line."

p.5

In next paragraph, Dedekind talks about how his contemporaries have tried to build the real numbers. He says "Instead of this, I demand that arithmetic shall be developed out of itself. … so we must endeavor completely to define the irrational numbers by means of the rational numbers alone."

In the third paragraph on p.5, Dedekind discusses how he has wrestled with what continuity means. Then he decides:

"In the preceding section, attention was called to the fact that every point *p* of the straight line produces a separation of the line into two portions such that every point of one portion lies to the left of every point of the other. I find the essence of continuity in the converse, i.e., in the following principle:

"If all the points of the straight line fall into two classes such that every point in the first class lies to the left of every point in the second class, then there exists one and only one point which produces this division of all the points into (these) two classes, this severing the straight line into two portions."

He then points out that this is just his idea for what continuity means. That is, he is going to take it as an "axiom" or definition of continuity. And he says, actual space might be different.

p.6

Part VI of Dedekind's article:
Now let's get down to business about how to build these new numbers.

As we have been discussing, we begin the set of rational numbers, R. We now create a set of objects $A_1|A_2$ where A_1, A_2 are subsets of R whose union is all of R and that have the property that every member a_1 of A_1 less than every member a_2 of A_2. He calls one of these objects a “Schnitt” in german and we’ll call one of them a “cut”.

What can these cuts look like? Think in small groups and then whole group

One kind of cut: Let a be a rational number. Define A_1 to be all the rational numbers less than a. Let A_2 be all the rational number that are greater than or equal to a. Is $A_1|A_2$ a cut?

Another example:
Let A_2 be the set of all positive rational numbers whose square is greater than 2.
Let A_1 be all other rational numbers.

Is $A_1|A_2$ a cut? Yes!

> If a is negative or 0, then clearly a is less than every member of A_2. If $a_1 > 0$ belongs to A_1, then a_1^2 is less than 2, and hence it is less than any a_2 whose square is greater than 2.
>
> Notice that A_1 has no greatest member and A_2 has no least member. (I omit the details of this proof. Among other ideas, it uses the fact that there is no rational number whose square equals two.)

p. 7

Returning to Dedekind, he notes “in the fact that not all cuts are produced by rational numbers consists the incompleteness or discontinuity of the domain of rational numbers.

“Whenever, then, we have to do with a cut $A_1|A_2$ produced by no rational number, we create a new, an irrational number α , which we regard as completely defined by this cut $A_1|A_2$; we shall say that the number α corresponds to this cut or that it produces this cut. From now on, therefore, to every definite cut, there corresponds a definite rational or irrational number, and we regard two numbers as different or unequal always and only when they correspond to essentially different cuts.”

So, following Dedekind, we define the set of real numbers to be the set $\mathfrak{R}$ of all cuts.

What is a real number? <-------------(Title of the talk !!)
Answer:
It is a Dedekind “cut” of the rational number line.

Now what remains is (the drudgery of) proving that the set $\mathfrak{R}$ of cuts satisfies the axioms of the real numbers.

What do we have to do?

Define equals and less than among cuts.
Define addition and multiplication

Here is a brief list of the steps involved:

Define the relations of "equals" and "less than" among cuts:
For two cuts α, β how do you decide whether $\alpha < \beta$ or $\alpha = \beta$ or $\beta < \alpha$.
p. 8
Trichotomy
When two cuts α, β are not equal, either $\alpha < \beta$ or $\beta < \alpha$.
p. 9
Completeness
The set $\mathfrak{R}$ possesses completeness, i.e. if the set $\mathfrak{R}$ of all real numbers breaks up into two classes, $\mathfrak{A}_1$, $\mathfrak{A}_2$ such that every member of $\mathfrak{A}_1$ is less than every member of $\mathfrak{A}_2$ then there exists one and only one number α of $\mathfrak{R}$ by which this separation is produced.

Part VI.

Addition

Let $\alpha = A_1|A_2$ and $\beta = B_1|B_2$ be cuts in $\mathfrak{R}$. We wish to define a cut $\gamma = \alpha + \beta$:

If c is any rational number, we put it in class C_1 if there are two numbers one a_1 in A_1 and another b_1 in B_1 such that their sum $a_1 + b_1$ is greater than c. We put all other rational numbers go in C_2. This is a separation of all the rational numbers into two classes C_1, C_2 that evidently forms a cut, since every number c_1 in C_1 is less than every number c_2 in C_2. (Needs proof.) Thus $\gamma = C_1|C_2$ is the sum $\alpha + \beta$.

I hope you get the idea.

Then one needs to set about proving the axioms for addition are satisfied using this definition.

Next you do the same for multiplication of cuts. Then you confirm all the multiplication axioms.

And so forth…

When you have completed confirming all the axioms listed in the appendix and proven them for our set of real numbers, $\mathfrak{R}$, you are done.

And so are we.

What questions that have arisen for you this morning?

This concludes the presentation.

There is another way to "build" the real numbers from the rational numbers. It involves creating ring of Cauchy sequences and then mod-ing out by the ideal comprised of the Cauchy sequences whose terms converge to zero.

APPENDIX

The real numbers form a **complete ordered field.** This means that the set R satisfies the following axioms:

The field axioms. R is a field; that is, it has two binary operations $(x, y) \to x + y$ and $(x, y) \to xy$ called **addition** and **multiplication** and two distinguished members 0 and 1, such that:
(A1) For all x, y ε R, $x + y = y + x$ (**commutativity**)
(A2) For all x, y, z ε R, $(x + y) + z = x + (y + z)$ (**associativity**)
(A3) For all x ε R, $x + 0 = x = 0 + x$ (0 **is an identity for addition**)
(A4) To each x ε R corresponds an element $-x$ ε R with $x + (-x) = 0 = (-x) + x$. (**additive inverse**)

(M1) For all x, y ε R, $xy = yx$, (**commutativity**)
(M2) For all x, y, z ε R $(xy)z = x(yz)$ (**associativity**)
(M3) 1 not= 0 and for all x ε R $x1 = x = 1x$ (1 **is an identity for multiplication**).
(M4) To each x ε R with x not= 0 there corresponds an x^{-1} ε R with $xx^{-1} = 1 = x^{-1}x$ (**multiplicative inverse**).

(DL) For all x, y, z ε R, $x(y + z) = xy + xz$ (**distributive law**).

The Order Axioms.
In addition to the above there is a relation < on R making it an **ordered field**, that is, it satisfies:
(O1) For all x, y ε R, exactly one of $x < y$, $x = y$, $y < x$ holds (**trichotomy**).
(O2) If $x < y$ and $y < z$, then $x < z$ (**transitivity**).
(O3) $x < y$ implies $x + z < y + z$. (**addition preserves order**)
(O4) $x < y$ and $z > 0$ implies $xz < yz$. (**multiplication by a number** > 0 **preserves order**).

The Completeness Axiom.
If A and B are non-empty subsets of R such that for all a ε A and b ε B, $a < b$, then there is an x ε R such that a <= x for all a ε A and x <= b for all b ε B. (Here x <= y is an abbreviation for $x < y$ or $x = y$.)

Notice that the rational number system satisfies all of these axioms except the last one.

ℜ

$A_1|A_2$