Solutions to the 1999 AP Calculus AB Exam Free Response Questions

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Problem 1.

a.

If $v[t] = t \sin t^2$ for $t \geq 0$, then when $t = 1.5$, we have

```
v[t_] = t Sin[t^2]

t Sin[t^2]
```

```
v[1.5]

1.16710979533
```

$v[1.5] > 0$. Velocity is positive, so motion is in the positive direction, or upward.

b.

Accleration at $t = 1.5$ is

```
v'[t]

2 t^2 Cos[t^2] + Sin[t^2]
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```
v'[1.5]

-2.04870810536
```

Acceleration at $t = 1.5$ is negative, so velocity is decreasing.

■ c.

By the Fundamental Theorem of Calculus, $y[t] = y[0] + \int_0^t v[t]\, dt = y[0] + \int_0^t \tau \sin[\tau^2]\, d\tau = \frac{5}{2} - \frac{1}{2}\cos[t^2]$. Thus, $y[2] = \frac{5}{2} - \frac{1}{2}\cos[4]$.

```
2.5 - 0.5 Cos[4.0]

2.82682181043
```

■ d.

Total distance traveled when $0 \le t \le 2$ is $\int_0^2 |v[t]|\, d\, dt = \int_0^2 |\tau \sin[\tau^2]|\, d\tau = \int_0^{\sqrt{\pi}} \tau \sin[\tau^2]\, d\tau - \int_{\sqrt{\pi}}^2 \tau \sin[\tau^2]\, d\tau$. Integrating numerically, we obtain

```
NIntegrate[τ Sin[τ^2], {τ, 0, √π}] -
 NIntegrate[τ Sin[τ^2], {τ, √π, 2}]

1.17317818957
```

Problem 2

■ a.

The area of the pictured region is $\int_{-2}^2 (4 - x^2)\, dx = (4x - \frac{1}{3}x^3) \Big|_{-2}^{2} = \frac{32}{3}$.

■ b.

Revolving the pictured region about the x-axis produces a solid whos volume is $\pi \int_{-2}^2 (16 - x^4)\, dx = \frac{256}{5}\pi$.

c.

$\pi \int_{-2}^{2} \left((k - x^2)^2 - (k - 4)^2\right) dx = \frac{256}{5} \pi.$

For the curious:

```
Solve[∫_{-2}^{2} ((k - x^2)^2 - (k - 4)^2) dx == 256/5, k]

{{k → 24/5}}
```

Problem 3.

First, let us assign appropriate values to the function *R*:

```
Map[Apply[(R[#1] = #2) &, #] &,
 {{0, 9.6}, {3, 10.4}, {6, 10.8}, {9, 11.2}, {12, 11.4},
  {15, 11.3}, {18, 10.7}, {21, 10.2}, {24, 9.6}}]

{9.6, 10.4, 10.8, 11.2, 11.4, 11.3, 10.7, 10.2, 9.6}
```

a.

```
Sum[R[k] 6, {k, 3, 21, 6}]

258.6
```

Approximately 258.6 gallons of water flows out of the pipe in the period $0 \le t \le 24$.

b.

The function R is given differentiable on $[0, 24]$, and it is also given that $R[0] = 9.6 = R[24]$. By Rolle's Theorem, there must be a $t \in (0, 24)$ such that $R'[t] = 0$.

c.

The average rate of flow is approximately $\frac{1}{24}\int_0^{24}\frac{1}{79}(768+23t-t^2)\,dt=\frac{852}{79}$ gallons per hour.

Problem 4

a.

An equation for the line tangent to the graph of f at the point where $x=0$ is $y=2-3x$.

b.

We are given that f has a continuous second derivative for all x. Consequently, f' is also continuous for all x. Because $f'[0]=-3$, $f'[x]<0$ for all x in some open interval, J, centered at 0. Consequently, f'' is decreasing throughout the interval J. Because $f''[0]=0$, this means that $f''[x]>0$ in that part of J which lies to the left of $x=0$ and that $f''[x]<0$ in that part of J which lies to the right of $x=0$. Because f'' changes sign at $x=0$, f has an inflection point at $x=0$.

c.

An equation for the line tangent to the graph of f at the point where $x=0$ is $y=g[0]+g'[0](x-0)=4+[e^0(3\cdot 2+2(-3))]\,x$, or $y=4$.

d.

$g'[x]=e^{-2x}(3f[x]+2f'[x])$, so
$g''[x]=[\frac{d}{dx}(e^{-2x})](3f[x]+2f'[x])+e^{-2x}[\frac{d}{dx}(3f[x]+2f'[x])]=$.
$-2e^{-2x}(3f[x]+2f'[x])+e^{-2x}(3f'[x]+2f''[x])=$
$-6e^{-2x}f[x]-4e^{-2x}f'[x]+3e^{-2x}f'[x]+2e^{-2x}f''[x]=e^{-2x}(-6f[x]-f'[x]+2f''[x]$

Hence $g''[0]=(-6)\cdot(2)-(-3)+2\cdot 0=-9$. By the Second Derivative Test, g has a local maximum at $x=0$.

Problem 5

■ a.

On the interval [2, 4], the graph is symmetric about the point (3, 0), so the integral over that interval is zero. Consequently, $\int_1^4 f[t]\,dt = \int_1^2 f[t]\,dt$ is the area of the trapezoid whose corners are (1, 0), (2, 0), (2, 1), and (1, 4), or $\frac{(4+1)}{2}\cdot 1 = \frac{5}{2}$. Thus, $g[4] = \frac{5}{2}$. $g[-2] = \int_1^{-2} f[t]\,dt = -\int_{-2}^1 f[t]\,dt$ is the negative of the area of a triangle with base 3 and height 4, or -6.

■ b.

By the Fundamental Theorem of Calculus, $g'[x] = \frac{d}{dx}\int_1^x f[t]\,dt = f[x]$. Hence $g'[1] = f[1] = 4$.

■ c.

The absolute minimum of $g[x]$ for $-2 \le x \le 4$ lies either at a point where $g'[x] = 0$ or where $x = -2$ or where $x = 4$. We have already seen, in part a above, that $g[-2] = -6$ and that $g[4] = \frac{5}{2}$. If $g'[x] = 0$, then, by our first observation in part b above, $f[x] = 0$. This happens only at $x = -2$, which we have already computed, and at $x = 3$. But f, and therefore g' undergoes a sign change from positive to negative as x increases through 3, so $x = 3$ must give a local maximum for g. It follows that g attains its local minimum of -6 at $x = -2$.

■ d.

If g is to have an inflection point somewhere, then g' must change from increasing to decreasing or from decreasing to increasing at that point. This happens when $x = 1$, but not when $x = 2$. Hence g has an inflection point at just one of the two points in question.

Problem 6

■ a.

If $y = x^{-2}$, then $y' = -2x^{-3}$. So the equation of a line tangent to the curve $y = \frac{1}{x^2}$ at the point $(w, \frac{1}{w^2})$ is $y = \frac{1}{w^2} - \frac{2}{w^3}(x - w) = -\frac{2}{w^3}x + \frac{3}{w^2}$. Hence $k = 3\,\frac{w}{2}$. When $w = 3$, this gives $k = \frac{9}{2}$.

■ b.

We saw in part a. that, in general, $k = \frac{3w}{2}$.

■ c.

From part a or part b, we have $k = 3\,\frac{w}{2}$. Hence, by implicit differentiation with respect to t, $\frac{dk}{dt} = \frac{3}{2}\,\frac{dw}{dt}$. If $\frac{dw}{dt} = 7$, then when $w = 5$ we must have $\frac{dk}{dt} = \frac{21}{2}$ units per second.

■ d.

The tangent line at $(w, \frac{1}{w^2})$ crosses the x-axis when $0 = -\frac{2}{w^3}\,x + \frac{3}{w^2}$, or when $x = \frac{3}{2}\,w$. The area of the triangle PQR is therefore $A = \frac{1}{2} \cdot (\frac{3}{2}\,w - w) \cdot \frac{1}{w^2} = \frac{1}{4\,w} = \frac{1}{4}\,w^{-1}$. Consequently, $\frac{dA}{dt} = -\frac{1}{4}\,w^{-2}\,\frac{dw}{dt}$. When $w = 5$ and $\frac{dw}{dt} = 7$, this gives $\frac{dA}{dt} = -\frac{1}{4} \cdot \frac{1}{25} \cdot 7 = -\frac{7}{100}$. Because this is negative, area is decreasing at this instant. (Note: Because w and $\frac{dw}{dt}$ are both positive at the critical instant, it suffices for solving this problem to notice that the differentiation produced a minus sign.)