

Solutions to the 2000 AP Calculus AB Exam Free Response Questions

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Problem 1.

■ a.

We must first find the point in the first quadrant where $e^{-x^2} = 1 - \cos x$. We'll call it b :

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b = x /. FindRoot [e-x2 == 1 - Cos [x], {x, 1}]
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0.94194408
```

The area between the two curves is $\int_0^b (e^{-x^2} - (1 - \cos x)) dx$. We integrate numerically:

```
NIntegrate [e-x2 - (1 - Cos [x]), {x, 0, b}]
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```
0.59096245
```

■ b.

By the method of washers, and integrating numerically, the volume generated when R is revolved about the x -axis is $\pi \int_0^b [(e^{-x^2})^2 - (1 - \cos x)^2] dx$:

```
.π NIntegrate [(e-x2)2 - (1 - Cos [x])2, {x, 0, b}]
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1.7466141
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■ **C.**

Integrating numerically, we obtain the volume of the solid described as $\int_0^b (e^{-x^2} - (1 - \cos x))^2 dx$.

$$\mathbf{NIntegrate} \left[\left(e^{-x^2} - (1 - \mathbf{Cos}[x]) \right)^2, \{x, 0, b\} \right]$$

0.46106351

Problem 2

■ **a.**

From the graph of runner A's velocity, which is given, we see that her velocity at time $t = 2$ is $\frac{20}{3}$ meters per second. Runner B's velocity at time t is given as $\frac{24t}{2t+3}$, so runner B's velocity at $t = 2$ is $\frac{48}{7}$ meters per second.

■ **b.**

Acceleration is the derivative, taken with respect to time, of velocity. In the case of runner A, at time $t = 2$, the slope of the velocity curve is $\frac{10}{3}$, so her acceleration at time $t = 2$ is $\frac{10}{3}$ meters per second per second.

$$\mathbf{D} \left[\frac{24 t}{2 t + 3}, t \right] /. t \rightarrow 2$$

$\frac{72}{49}$

Runner B's acceleration at $t = 2$ is $\frac{72}{49}$ meters per second per second.

■ **C.**

Distance travelled is the integral of velocity. Hence, reasoning from the graph of runner A's velocity, we find that runner A covered $\frac{1}{2} \cdot 3 \cdot 10 + 7 \cdot 10 = 85$ meters over the interval $0 \leq t \leq 10$.

$$\int_0^{10} \frac{24t}{2t+3} dt$$

$$6 (20 - 3 \operatorname{Log}[23] + \operatorname{Log}[27])$$

N [%]

83.336125

Runner *B* covered $6 (20 - 3 \operatorname{Log}[23] + \operatorname{Log}[27])$ meters, or about 83.336 meters, in the same time interval.

Problem 3.

■ a.

According to the First Derivative Test, a differentiable function attains a relative minimum at a point where its derivative changes sign from negative to positive as the independent variable increases. There is just one such point for the derivative whose graph is shown: $x = -1$. Consequently, f has a relative minimum at $x = -1$.

■ b.

According to the First Derivative Test, a differentiable function attains a relative maximum at a point where its derivative changes sign from positive to negative as the independent variable increases. There is just one such point for the derivative whose graph is shown: $x = -5$. Consequently, f has a relative maximum at $x = -5$.

■ c.

The second derivative is negative throughout intervals where f' has a tangent line that slopes downward to the right. There are three such intervals: $(-7, -3)$, $(2, 3)$, and $(3, 5)$. (Note that the vertical tangent at $x = 3$ is a point where f'' is undefined.)

■ d.

By the Fundamental Theorem of Calculus, $f[t] = \int_{-7}^t f'[x] dx$. Thus, $f[t]$ gives the algebraic sum of the signed area that is bounded by the x -axis, the curve $y = f'[x]$, and the vertical lines $x = -7$ and $x = t$. (We observe the usual convention that area above the horizontal axis is positive area, while area below the horizontal axis is negative area.) It is evident from the picture that the maximal such area, taking signs into account, is that for which $t = 7$, so the absolute maximum for $f[t]$ occurs at $t = 7$.

Problem 4

■ a.

Water leaks out of the tank at the rate of $\sqrt{t+1}$ gallons per minute, so the tank loses

$$\int_0^3 \sqrt{t+1} \, dt = \frac{2}{3} (t+1)^{3/2} \Big|_0^3 = \frac{14}{3} \text{ gallons during the interval } 0 \leq t \leq 3.$$

■ b.

During the interval $0 \leq t \leq 3$, a total of $\frac{14}{3}$ gallons of water have leaked from the tank, while $3 \cdot 8 = 24$ gallons have entered the tank. Because there were 30 gallons of water in the tank at time $t=0$, the tank contains $30 + 24 - \frac{14}{3} = \frac{148}{3}$ gallons when $t = 3$.

■ c.

The amount $A[t]$ of water in the tank at time t is $A[t] = 30 + 8t - \int_0^t \sqrt{\tau+1} \, d\tau$.

■ d.

We have $A'[t] = 8 - \sqrt{t+1}$. This vanishes when $t = 63$, is positive for $0 < t < 63$, and is negative for $63 < t < 120$. It follows that $A[t]$ is maximal when $t = 63$, because A is increasing on $[0, 63]$ and decreasing on $[63, 120]$.

Problem 5

■ a.

Differentiating both sides of the equation $xy^2 - x^3y = 6$ implicitly, we obtain $y^2 + 2xyy' - 3x^2y - x^3y' = 0$, whence $(2xy - x^3)y' = 3x^2y - y^2$. Dividing both sides of the latter equation by $2xy - x^3$, which we must assume to be non-zero, we obtain $y' = \frac{3x^2y - y^2}{2xy - x^3}$, as desired.

■ b.

Setting $x = 1$ in the original equation, we obtain $y^2 - y = 6$, whence $y = 3$ or $y = -2$. There are therefore two points with x -coordinate 1; they are $(1, 3)$ and $(1, -2)$. At $(1, 3)$, we have $y' = \frac{3 \cdot 1^2 \cdot 3 - 3^2}{2 \cdot 1 \cdot 3 - 1^3} = \frac{0}{5} = 0$, so that an equation of the line tangent to the curve at $(1, 3)$ is $y = 3$. At $(1, -2)$, we have $y' = \frac{3 \cdot 1^2 \cdot (-2) - (-2)^2}{2 \cdot 1 \cdot (-2) - 1^3} = \frac{-10}{-5} = 2$. An equation of the line tangent to the curve at $(1, -2)$ is therefore $y = -2 + 2(x - 1)$, or $y = 2x - 4$.

■ c.

We treat y as the independent variable in the original equation, and letting the prime denote differentiation with respect to y , implicit differentiation now gives us $x' y^2 + 2xy - 3x^2 x' y - x^3 = 0$, whence $x' = \frac{x^3 - 2xy}{y^2 - 3x^2 y}$. At a point with a vertical tangent, x' must vanish, so we must have $x^3 - 2xy = 0$. Thus, $x = 0$ or $y = \frac{1}{2}x^2$. But $xy^2 - x^3 y = 6$, so $x = 0$ is not possible. On the other hand, substituting $y = \frac{1}{2}x^2$ in the original equation yields $\frac{1}{4}x^5 - \frac{1}{2}x^5 = 6$, or $x^5 = -24$. There is thus a vertical tangent at the point $(-\sqrt[5]{24}, \sqrt[5]{18})$.

Problem 6

■ a.

If $\frac{dy}{dx} = \frac{3x^2}{3^2 y}$, then $\int_0^x e^{2y[\xi]} y'[\xi] d\xi = \int_0^x 3\xi^2 d\xi$, or $\frac{1}{2} e^{2y[\xi]} \Big|_0^x = \xi^3 \Big|_0^x$. Thus, $\frac{1}{2} e^{2y[x]} - \frac{1}{2} e^{2y[0]} = x^3$, or $y[x] = \frac{1}{2} \ln(2x^3 + e^{2y[0]})$. Our initial condition $y[0] = \frac{1}{2}$ therefore leads us to $f[x] = \frac{1}{2} \ln(2x^3 + e)$.

■ b.

The domain of the function f of part a) above is the set $\{x \in \mathbb{R} : 2x^3 + e > 0\} = \{x \in \mathbb{R} : x > -\sqrt[3]{e/2}\}$. Because the equation $v = 2x^3 + e$ has a solution for every $v > 0$, the range of f is \mathbb{R} .