Solutions to the 2001 AP Calculus AB Exam

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1.

▪ a.

Let's solve the equation $2 - x^3 = \text{Tan}[x]$ numerically and call the solution b:

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FindRoot[2 - x^3 == Tan[x], {x, 1.0}]
```

```
{x → 0.90215508}
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b = x /. %[[1]]
```

```
0.90215508
```

Here is the area of the region R:

$$\int_0^b \text{Tan}[x]\, dx + \int_b^{2^{1/3}} (2 - x^3)\, dx$$

```
0.72933733
```

▪ b.

Here is the area of the region S:

$$\int_0^b (2 - x^3 - \text{Tan}[x])\, dx$$

```
1.1605442
```

▪ c.

And, using the "washer" method, the area of the volume generated by revolving the region S about the x-axis:

$$\pi \int_0^b \left((2 - x^3)^2 - \text{Tan}[x]^2\right) dx$$

```
8.3318163
```

2.

■ a.

We have $W'[12] \simeq \frac{W[15]-W[9]}{6} = \frac{21-24}{6} = -\frac{1}{2}$ degrees Celsius/day.

■ b.

Using the Trapezoid Rule $A \simeq \frac{1}{2}\,(f[x_0] + 2\,f[x_1] + \ldots + 2\,f[x_{n-1}] + f[x_n])\,\Delta x$, the required average value is about:

$$\frac{1}{15}\;\frac{20 + 2\;31 + 2\;28 + 2\;24 + 2\;22 + 21}{2}\;3$$

$$\frac{251}{10}$$

The average water temperatureis about 25.100 degrees Celsius.

■ c.

If $P[t]$ is given by

P[t_] = 20 + 10 t $e^{-t/3}$

$20 + 10\, e^{-t/3}\, t$

then P'[t] is

P'[t]

$10\, e^{-t/3} - \frac{10}{3}\, e^{-t/3}\, t$

Putting $t = 12$, we have

P'[12]

$-\frac{30}{e^4}$

N[%]

-0.54946917

On the 12th day, the water temperature is decreasing at a rate of about −0.549 degrees Celsius per day.

■ d.

The required average value is given by

```
1/15 ∫_0^15 P[t] ⅆt
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$$\frac{1}{15}\left(90+\frac{10\,(-54+30\,e^5)}{e^5}\right)$$

```
Expand[%]
```

$$26-\frac{36}{e^5}$$

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N[%]
```

25.757434

Average temperature over $0 \le t \le 18$ is therefore about 25.757 degrees Celsius.

3.

■ a.

When $t = 2$, the graph shows that acceleration is 15 ft./sec. This is a positive number, so velocity is increasing when $t = 2$.

■ b.

The portion of the acceleration curve on the interval $6 \le t \le 12$ is symmetric about the point (6, 0) with the portion of the acceleration curve on the interval $0 \le t \le 6$. Consequently the integral of acceleration from 0 to 12 (which is total change in velocity over that interval) is zero. Thus, velocity is 55 ft./sec. when $t = 12$ sec.

■ c.

The car's absolute maximum velocity for $0 \le t \le 18$ is 115 ft./sec., which is the velocity it attains when $t = 6$ sec.. Velocity increases from 55 ft./sec. as long as acceleration is positive—that is, until $t = 6$. Thereafter it decreases as long as acceleration is negative—that is, while $6 \le t \le 14$. Finally, it increases again while $14 \le t \le 18$. However, the area under the acceleration curve on the latter interval is less than the area between the acceleration curve and the t-axis on the interval $6 \le t \le 14$, so the total increase in velocity that accrues while $14 \le t \le 18$ is not enough to balance out the total decrease that accrued while $6 \le t \le 14$. This means that velocity attains it absolute maximum for $0 \le t \le 18$ when $t = 6$ sec. We calculate this maximum value by finding the area of the trapezoid over the interval $0 \le t \le 6$, which is $\frac{1}{2}(2+6)\cdot 15 = 60$, and adding it to the initial velocity, 55 ft / sec to obtain a maximal velocity of 115 ft/sec.

■ d.

The car never reaches a velocity of 0 ft./sec. In fact, the absolute minimum velocity attained by the car occurs when $t = 16$, and this velocity is the initial velocity of 55 ft./sec. plus the area of the region above the t-axis over the interval [0, 6] minus the area of the region below the t-axis over the inverval [6, 16], which is $55 + 60 - 105 = 10$ ft./sec.

4.

■ a.

If $h'[x] = \frac{x^2-2}{x} = \frac{(x-\sqrt{2})(x+\sqrt{2})}{x}$, then $h'[x] = 0$ when $x = \pm\sqrt{2}$, so the graph of h has a horizontal tangent when $x = \pm\sqrt{2}$. Note that $(x+\sqrt{2})$ changes sign from negative to positive as x increases through $x = -\sqrt{2}$, while $(x-\sqrt{2})$ and x are both negative near $x = -\sqrt{2}$. Hence $h'[x]$ changes sign from negative to positive as x increases through $-\sqrt{2}$. This means that $x = -\sqrt{2}$ give a local minimum for the graph of h. On the other hand $(x-\sqrt{2})$ changes sign from negative to positive as x increases through $\sqrt{2}$, where both x and $(x+\sqrt{2})$ are positive. Hence $h'[x]$ changes sign from negative to positive as x increases through $\sqrt{2}$, and this means that h has a local minimum at $x = \sqrt{2}$ also.

■ b.

We have $h''[x] = \frac{d}{dx}(x - 2x^{-1}) = 1 + \frac{2}{x^2}$, which is always positive (except, of course, where $x = 0$). Hence h is concave upward on $(-\infty, 0)$ and on $(0, \infty)$.

■ c.

The equation of the tangent line to the graph of h at $x = 4$ is $y = h[4] + h'[4](x-4)$, or $y = (-3) + \frac{(4^2-2)}{4}(x-4)$. This is $y = \frac{7}{2}x - 17$.

■ d.

We have $h''[x] = 1 + \frac{2}{x^2}$, so that $h''[x] > 1$
for all $x > 4$.

This means that $h'[x] > h'[4] = 7/2$ for all $x > 4$.
Consequently,

$$h[x] - h[4] = \int_4^x h'[t]\,dt > \int_4^x \frac{7}{2}\,dt = \frac{7}{2}(x-4)$$

for $x > 4$. Consequently, when $x > 4$, we have $h[x] > \frac{7}{2}(x-4) + h[4] = \frac{7}{2}x - 17$, and the latter expression is the right hand side of the equation of the tangent line as found above. This means that the line tangent to the graph of $y = h[x]$ at $x = 4$ lies above the graph of h for $x > 4$.

5.

■ a.

We have $f[x] = 4x^3 + ax^2 + bx + k$, so $f'[x] = 12x^2 + 2ax + b$, and $f''[x] = 24x + 2a$. Because there must be an inflection point at $x = -2$, it follows that $0 = f''[-2] = -48 + 2a$. From this we see that $a = 24$. Thus, $f'[x] = 12x^2 + 48x + b$, and because of the local minimum at $x = -1$, we conclude that $0 = f'[-1] = 12(-1)^2 + 48(-1) + b = -36 + b$. Hence $b = 36$.

■ b.

From part a, above, we have $f[x] = 4x^3 + 24x^2 + 36x + k$, and so $32 = \int_0^1 f[x]\,dx = (x^4 + 8x^3 + 18x^2 + kx)\,|_0^1 = 27 + k$. Hence $k = 5$.

6.

■ a.

We have $y' = y^2(6 - 2x)$, so $y'' = 2yy'(6 - 2x) - 2y^2$. Because the point $(3, \frac{1}{4})$ lies on the graph of $y = f[x]$, we have $y'[3] = (\frac{1}{4})^2 (6 - 2 \cdot 3) = 0$. Consequently, $y''[3] = -\frac{1}{8}$.

■ b.

If $y' = y^2(6 - 2x)$, with $y[3] = \frac{1}{4}$, then $y^{-2}\,dy = (6 - 2x)\,dx$, and we have $\int_{1/4}^{y} v^{-2}\,dv = \int_3^x (6 - 2u)\,du$. Hence $-v^{-1}\,|_{1/4}^{y} = (6u - u^2)\,|_3^x$, or $\frac{-1}{y} + 4 = (6x - x^2) - 9$. Thus, $y = \frac{1}{13 - 6x + x^2}$.