Solutions to the 2002 AP Calculus AB Exam

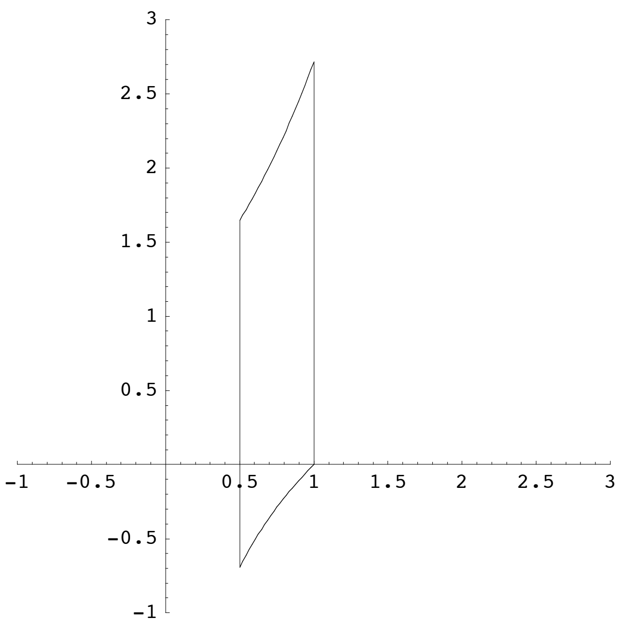
Louis A. Talman
Metropolitan State College of Denver

1.

■ a.

Here is the region between the graphs on the specified interval:

```
p1 = Plot[{ⅇ^x, Log[x]}, {x, 1/2, 1}, PlotRange → {{-1, 3}, {-1, 3}},
  AspectRatio → Automatic, DisplayFunction → Identity];
p2 = Graphics[{Line[{{1/2, ⅇ^(1/2)}, {1/2, Log[1/2]}}], Line[{{1, ⅇ}, {1, 0}}]}];
Show[p1, p2, DisplayFunction → $DisplayFunction]
```




```
- Graphics -
```

The area is given by

$$\int_{1/2}^{1} (e^x - \mathrm{Log}[x])\, dx$$

$$\frac{1}{2}\left(1 - 2\sqrt{e} + 2e - \mathrm{Log}[2]\right)$$

And here, for the curious, is a numeric answer:

```
N[%]
```

1.222987

■ b

Using the method of washers, we find that the required volume is

$$\pi \int_{1/2}^{1} ((4 - \text{Log}[x])^2 - (4 - e^x)^2)\, dx$$

$$\frac{1}{2}\pi\left(10 - 16\sqrt{e} + 17\,e - e^2 - 10\,\text{Log}[2] - \text{Log}[2]^2\right)$$

(Use integration by parts twice in succession to find $\int(\ln x)^2\, dx$.) Numerically, this is

```
N[%]
```

23.609493

■ c.

Let

$$h[x\_] = e^x - \text{Log}[x]$$

$$e^x - \text{Log}[x]$$

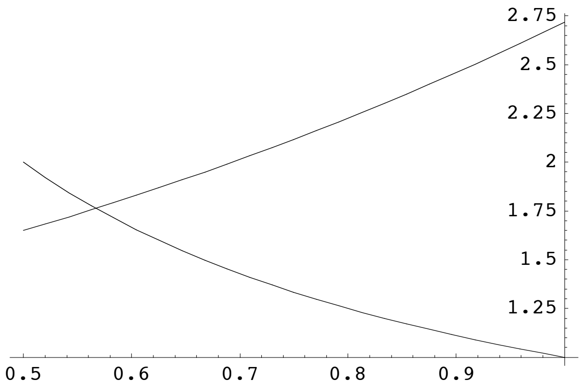
We seek the critical pointsof h that lie in the interval $(\frac{1}{2}, 1)$ and points in the same interval where the derivative is undefined.

```
h'[x]
```

$$e^x - \frac{1}{x}$$

This is defined everywhere in $(\frac{1}{2}, 1)$, so we consider only critical points. Examination of a graph

```
Plot[{e^x, 1/x}, {x, 1/2, 1}]
```




```
- Graphics -
```

indicates that there is exactly one critical point. We find it numerically:

```
b = x /. FindRoot[h'[x] == 0, {x, 0.75}]

0.56714319
```

The points of interest to us are the endpoints and the critical point, because both the absolute maximum and the absolute minimum must occur at such points. Thus we evaluate *h* at each of the points 0.5, 0.567143, and 1.0.

```
Map[h, {0.5, b, 1.0}]

{2.3418685, 2.3303661, 2.7182818}
```

The largest of these outputs is the number e itself, so it is the maximum value taken on by h on $[\frac{1}{2}, 1]$. The smallest output is 2.33037, so it is the minimum value taken on by h on $[\frac{1}{2}, 1]$.

2.

■ a.

We are given

$$\text{EnterRate[t\_]} = \frac{15600}{t^2 - 24\,t + 160}$$

$$\frac{15600}{160 - 24\,t + t^2}$$

$$\text{LeaveRate[t\_]} = \frac{9890}{t^2 - 38\,t + 370}$$

$$\frac{9890}{370 - 38\,t + t^2}$$

From these and the fact that there are no people in the park at 9:00, we have

```
NumberEntered[t_] = ∫_9^t EnterRate[τ] dτ
```

$$15600\left(\frac{1}{4}\,\text{ArcTan}\left[\frac{3}{4}\right]+\frac{1}{4}\,\text{ArcTan}\left[\frac{1}{4}\,(-12+t)\right]\right)$$

Consequently, at 5:00 pm,

```
NumberEntered[17.0]
```

```
6004.2703
```

so that 6004 people have entered the park by 5:00 pm.

■ b.

Revenue is given by

```
15 NumberEntered[17] + 11 (NumberEntered[23] - NumberEntered[17])
```

$$234000\left(\frac{1}{4}\,\text{ArcTan}\left[\frac{3}{4}\right]+\frac{1}{4}\,\text{ArcTan}\left[\frac{5}{4}\right]\right)+ 11\left(-15600\left(\frac{1}{4}\,\text{ArcTan}\left[\frac{3}{4}\right]+\frac{1}{4}\,\text{ArcTan}\left[\frac{5}{4}\right]\right)+15600\left(\frac{1}{4}\,\text{ArcTan}\left[\frac{3}{4}\right]+\frac{1}{4}\,\text{ArcTan}\left[\frac{11}{4}\right]\right)\right)$$

```
N[%] // InputForm
```

```
104048.16522947158
```

Rounded to the nearest dollar, as required, this gives $104,048.

■ c.

If $H[t] = \int_9^t (\text{EnterRate}[x] - \text{LeaveRate}[x])\,dx$, then $H'[t] = \text{EnterRate}[t] - \text{LeaveRate}[t]$. (This is just the Fundamental Theorem of Calculus.) Hence $H'[17]$ is given by

```
EnterRate[17] - LeaveRate[17]
```

$$-\frac{202690}{533}$$

or, numerically,

```
N[%]
```

```
-380.28143
```

$H[t]$ gives the number of people in the park at time t, where $9 \le t \le 23$. Thus, $H[17] = 3725$ is the number of people in the park at 5:00 pm. $H'[t]$ gives the rate at which the number of people in the park is increasing at time t, again for $9 \le t \le 23$. $H'[17] = -380$ means that at 5:00 pm people are leaving the park at a rate of 380 people per hour.

▪ d.

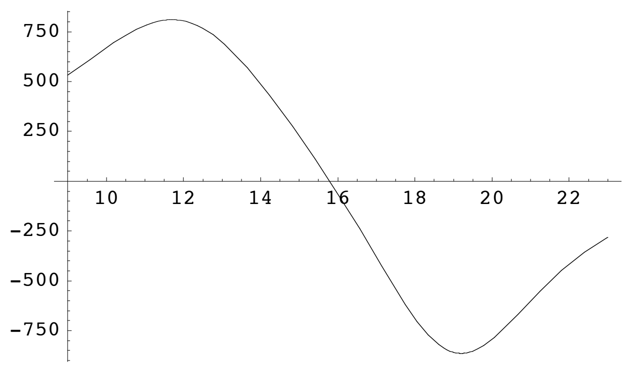
We have an expression for $H'[t]$:

```
HPrime[t_] = EnterRate[t] - LeaveRate[t] //
  Together
```

$$\frac{10\ (418960 - 35544\ t + 571\ t^2)}{(370 - 38\ t + t^2)\ (160 - 24\ t + t^2)}$$

Let's look at a picture to see if surprises are likely:

```
Plot[HPrime[t], {t, 9, 23}]
```



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Looks pretty simple; the maximum should occur around 4 o'clock in the afternoon. Let's nail it:

```
Solve[HPrime[t] == 0, t]
```

$$\left\{\left\{t \to \frac{4}{571}\ (4443 - \sqrt{4788614})\right\},\ \left\{t \to \frac{4}{571}\ (4443 + \sqrt{4788614})\right\}\right\}$$

```
N[%]
```

{{t → 15.794815}, {t → 46.453872}}

The second root is outside the region of interest, but the first one is right where we thought it should be. The model therefore predicts that the number of people in the park is a maximum when $t = \frac{4}{571}\left(4443 - \sqrt{4788614}\right)$.

3.

▪ a.

We are given

```
v[t_] = Sin[π/3 t]
```

$\text{Sin}\left[\frac{\pi t}{3}\right]$

```
x0 = 2
```

2

Acceleration is $v'[t]$:

```
v'[t]
```

$\frac{1}{3}\pi\,\text{Cos}\left[\frac{\pi t}{3}\right]$

```
% /. t → 4
```

$-\frac{\pi}{6}$

Acceleration at $t = 4$ is $\frac{-\pi}{6}$. (The problem gives no units, so we give none.)

■ b.

Note that if $3 < t < 4.5$, then $\pi < \frac{\pi t}{3} < \frac{3\pi}{2}$, and we know that the cosine function is negative on $(\pi, \frac{3\pi}{2})$. Because $v'[t]$ is a positive constant times $\cos(\frac{\pi t}{3})$, we conclude that $v'[t]$ is negative on $(3, 4.5)$. This means that $v[t]$ is decreasing on $(3, 4.5)$. Thus, Statement I is correct.

On the other hand, the sine function takes on negative values in the interval $(\pi, \frac{3\pi}{2})$, so if $s[t]$ denotes speed, then $s[t] = -\sin(\frac{\pi}{3} t) = -v[t]$ for $3 < t < 4.5$. Thus, $s'[t] = -v'[t]$ on that interval, and so $s'[t] > 0$ on $(3, 4.5)$. This means that speed increases throughout $(3, 4.5)$. Statement II is also correct.

■ c.

Total distance is the integral of speed, or $\int_0^4 s[\tau]\, d\tau$, in this case. Because $v[t] = \sin(\frac{\pi}{3} t)$ undergoes a sign-change at $t = 3$, we have total distance given by:

$$\int_0^3 v[\tau]\, d\tau - \int_3^4 v[\tau]\, d\tau$$

$\frac{15}{2\pi}$

Numerically, this is

```
N[%]
```

2.3873241

■ d.

Letting $x[t]$ denote position at time t, we have, by the Fundamental Theorem of Calculus, $x[t] = x[0] + \int_0^t v[\tau]\, d\tau$. We were given that $x[0] = x_0 = 2$. Thus, position at time t is

$$2 + \int_0^t v[\tau]\, d\tau$$

$$2 + \frac{3}{\pi} - \frac{3 \operatorname{Cos}[\frac{\pi t}{3}]}{\pi}$$

Setting t equal to 4 in this expression yields

```
% /. t → 4
```

$$2 + \frac{9}{2\pi}$$

Numerically, this is

```
N[%]
```

```
3.4323945
```

4.

■ a.

The function f, given pictorially, can be written as $f[x] = -3 + 3(x+2) = 3x + 3$ for $-2 \le x \le 0$; $f[x] = -3x + 3$ for $0 \le x \le 2$. Then $g[x] = \int_0^x (3t+3)\, dt = 3x + \frac{3}{2}x^2$ when $-2 \le x \le 0$, and $g[x] = \int_0^x (-3t+3)\, dt = 3x - \frac{3}{2}x^2$ when $0 < x < 2$. Thus, $g[-1] = -3 + \frac{3}{2} = \frac{-3}{2}$. (We could also have found this by looking at the area of the appropriate triangle in the figure.) By the Fundamental Theorem of Calculus, $g'[-1] = f[-1] = 0$. Also by the Fundamental Theorem of Calculus, $g''[-1] = f'[-1] = 3$.

■ b.

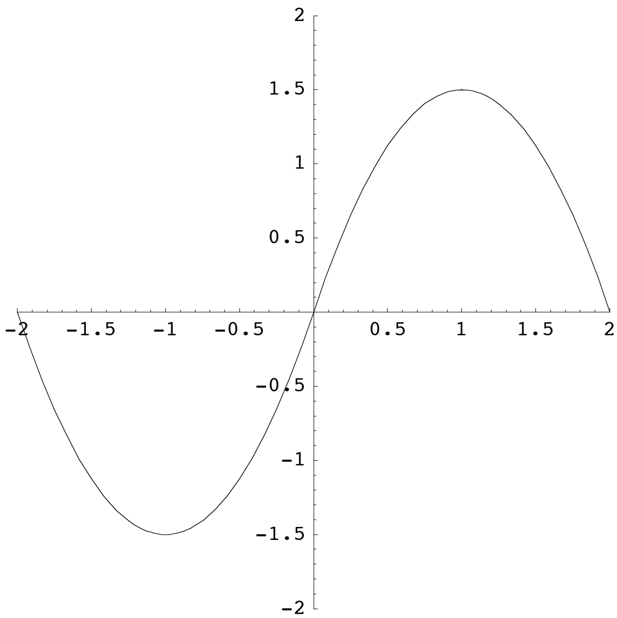
As we noted above for the special case $t = -1$, the Fundamental Theorem of Calculus tells us that $g'[t] = f[t]$. Thus, g is increasing on the closures of those intervals where $f[t] > 0$. From the picture, we conclude that g is an increasing function on the interval $[-1, 1]$, and on that interval only.

■ c.

From $g'[t] = f[t]$, which we derived above, we conclude that $g''[t] = f'[t]$. Now $f'[t] = 3$ for $-2 < t < 0$, and $f'[t] = -3$ for $0 < t < 2$. We conclude that g is concave downward on the interval $(0, 2)$, where $f'[t] < 0$.

▪ d.

```
p1 = Plot[3 x + 3/2 x^2, {x, -2, 0}, DisplayFunction → Identity];
p2 = Plot[3 x - 3/2 x^2, {x, 0, 2}, DisplayFunction → Identity];
Show[p1, p2, PlotRange → {{-2, 2}, {-2, 2}},
 AspectRatio → Automatic, DisplayFunction → $DisplayFunction]
```



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5.

▪ a.

When $h = 5$, we have $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{5}{2}\right)^2 5 = \frac{125\pi}{12}$ cm^3. (By similar triangles, $r = \frac{h}{2}$.)

▪ b.

For all t we have $V[t] = \frac{1}{3}\pi r[t]^2 h[t] = \frac{1}{12}\pi h[t]^3$. Thus $V'[t] = \frac{1}{4}\pi h[t]^2 h'[t] = -\frac{3}{40}\pi h[t]^2$. This gives $V'[5] = -\frac{15}{8}\pi \frac{\text{cm}^3}{\text{hr}}$.

■ c.

As we noted above, $h[t] = 2\,r[t]$, by similar triangles. Thus, $V'[t] = -\frac{3}{40}\,\pi\,h[t]^2 = -\frac{3}{10}\,\pi\,r[t]^2$. But $A[t] = \pi\,r[t]^2$. Thus, $V'[t] = -\frac{3}{10}\,A[t]$, so that $V'[t]$ is proportional to $A[t]$ with constant of proportionality $-\frac{3}{10}$.

6.

■ a.

By the Fundamental Theorem of Calculus, $\int_0^{1.5}(3\,f'[x] + 4)\,dx = (3\,f[x] + 4\,x)\,|_0^{1.5} = (3\,f[1.5] + 4\,(1.5)) - (3\,f[0] + 4\,(0)) = (-3+6)-(-21+0)$. Thus, the value of the integral is 24.

■ b.

The equation of the tangent line at (x_0, y_0) is $y = y_0 + f'[x_0]\,(x - x_0)$, so the required line has equation $y = -4 + 5\,(x-1)$, or $y = 5\,x - 9$. We obtain an approximate value for $f[1.2]$ by substituting $x = 1.2$ into the equation for the tangent line: $f[1.2] \sim 5\,(1.2) - 9 = -3$. We have been given that $f''[x] > 0$ everywhere in the region of interest, so we know that the curve is concave upward in that region. This means that the tangent line at any point in the region lies below the curve throughout the region. Consequently, our estimate of -3 for $f[1.2]$ is an underestimate: $f[1.2] > -3$.

■ c.

We are given that $f''[x]$ exists throughout $[-1.5,\ 1.5]$. Consquently, f' is continuous throughout $[-1.5, 1.5]$. By the Mean Value Theorem, there must be a number c strictly between 0 and 0.5 such that $f''[c]\,(0.5 - 0) = f'[0.5] - f'[0]$, Hence $f''[c] = (f'[0.5] - f'[0])/0.5 = 6$. The required number r is therefore 6.

■ d.

It is not possible that f and g are the same. As we have seen in part c. above, f' must be continuous throughout $[-1.5,\ 1.5]$ because we have been given the existence of $f''[x]$ in that interval. However, $g'[x] = 4\,x - 1$ when $x < 0$, and $g'[x] = 4\,x + 1$ when $x > 0$. It follows that g' is not continuous at $x = 0$.

Note: It also follows from these observations that $g'[0]$ is not defined. Derivatives have the Intermediate Value Property, and so cannot have jump discontinuities. This is not on the AB syllabus; it is, however, possible to see directly that $g'[0]$ is undefined.