Solutions to the 2002 AP Calculus AB Exam (Form B)

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1.

■ a.

We first find the intersection point for the two curves:

$$\mathbf{Solve}\left[4-2x==\frac{x^3}{1+x^2},\ x\right]$$

$$\left\{\left\{x\to\frac{4}{9}-\frac{(1+i\sqrt{3})\,(221+27\sqrt{67})^{1/3}}{9\;2^{2/3}}+\frac{1-i\sqrt{3}}{9\,(2\,(221+27\sqrt{67}))^{1/3}}\right\},\right.$$
$$\left\{x\to\frac{4}{9}-\frac{(1-i\sqrt{3})\,(221+27\sqrt{67})^{1/3}}{9\;2^{2/3}}+\frac{1+i\sqrt{3}}{9\,(2\,(221+27\sqrt{67}))^{1/3}}\right\},$$
$$\left.\left\{x\to\frac{4}{9}-\frac{2^{2/3}}{9\,(221+27\sqrt{67})^{1/3}}+\frac{1}{9}\,(2\,(221+27\sqrt{67}))^{1/3}\right\}\right\}$$

We want the real root, of course. Let's call it *b*:

```
b = x /. %[[3]]
```

$$\frac{4}{9}-\frac{2^{2/3}}{9\,(221+27\sqrt{67})^{1/3}}+\frac{1}{9}\,(2\,(221+27\sqrt{67}))^{1/3}$$

```
N[b]
```

```
1.4876644
```

That's reasonable...

The area of the specified region is

$$\int_0^b \left(4-2\,x-\frac{x^3}{1+x^2}\right) dx$$

$$\frac{1}{18\,\left(221+27\sqrt{67}\right)^{2/3}}\left(1178\;2^{1/3}+144\;2^{1/3}\sqrt{67}-79\;2^{2/3}\left(221+27\sqrt{67}\right)^{1/3}-9\;2^{2/3}\sqrt{67}\left(221+27\sqrt{67}\right)^{1/3}+28\left(221+27\sqrt{67}\right)^{2/3}+9\left(221+27\sqrt{67}\right)^{2/3}\,\mathrm{Log}\Big[\frac{1}{27\left(221+27\sqrt{67}\right)^{2/3}}\left(590\;2^{1/3}+72\;2^{1/3}\sqrt{67}+71\;2^{2/3}\left(221+27\sqrt{67}\right)^{1/3}+9\;2^{2/3}\sqrt{67}\left(221+27\sqrt{67}\right)^{1/3}+31\left(221+27\sqrt{67}\right)^{2/3}\right)\Big]\right)$$

This is unhelpful; a numeric answer would be more informative:

```
% // N
```

```
3.2145647
```

The required area is 3.214.

■ b

The method of washers is appropriate here. The required volume is

$$\pi\int_0^b \left((4-2\,x)^2-\left(\frac{x^3}{1+x^2}\right)^2\right) dx \;\mathbf{//\;Expand\;//\;Simplify}$$

$$\Bigg(\pi\Bigg(\left(221+27\sqrt{67}\right)^{5/3}\left(22748-4185\,\mathrm{ArcTan}\Big[\frac{1}{9}\left(4-\frac{2^{2/3}}{\left(221+27\sqrt{67}\right)^{1/3}}+\left(442+54\sqrt{67}\right)^{1/3}\right)\Big]\right)+2^{2/3}\left(221+27\sqrt{67}\right)^{1/3}\Big(17785643+2172861\sqrt{67}-270\left(15986+1953\sqrt{67}\right)\mathrm{ArcTan}\Big[\frac{1}{9}\left(4-\frac{2^{2/3}}{\left(221+27\sqrt{67}\right)^{1/3}}+\left(442+54\sqrt{67}\right)^{1/3}\right)\Big]\Big)+2\;2^{1/3}\Big(99206794+12120039\sqrt{67}-135\left(130319+15921\sqrt{67}\right)\mathrm{ArcTan}\Big[\frac{1}{9}\left(4-\frac{2^{2/3}}{\left(221+27\sqrt{67}\right)^{1/3}}+\left(442+54\sqrt{67}\right)^{1/3}\right)\Big]\Big)\Bigg)\Bigg)\Big/\left(54\left(221+27\sqrt{67}\right)\left(2^{2/3}\left(71+9\sqrt{67}\right)\left(221+27\sqrt{67}\right)^{1/3}+31\left(221+27\sqrt{67}\right)^{2/3}+2\;2^{1/3}\left(295+36\sqrt{67}\right)\right)\right)$$

```
% // N
```

```
31.88487
```

The required volume is 31.885.

■ c.

The area of the cross-section at x is $\left((4-2x)-\frac{x^3}{1+x^2}\right)^2$. Thus, the required volume is

```
NIntegrate[(4 - 2 x - x^3/(1 + x^2))^2, {x, 0, b}]
```

```
8.9970039
```

The required volume is 8.997.

Note: The symbolic integral is such a mess that we didn't even bother to display it. We did a numeric integration instead.

2.

■ a.

We are given

```
P'[t_] = 1 - 3 ⅇ^(-0.2 √t)
```

```
1 - 3 ⅇ^(-0.2 √t)
```

```
P'[9]
```

```
-0.64643491
```

Now $P'[9] < 0$, so the amount of pollutant in the lake is decreasing when $t = 9$.

■ b.

Because the exponential $e^{-0.2\sqrt{t}}$ decreases from 1 to 0 on $0 \le t < \infty$, $P'[t]$ is an increasing function on that interval. Moreover, $P'[0] = -2$ and $\lim_{t\to\infty} P'[t] = 1$. Because P' is a continuous, strictly increasing function on $0 \le t < \infty$, it follows P' has exactly one zero, and that P therefore has exactly one critical point, in the interval--where, moreover, P' changes sign from negative to positive. Thus, this zero of P' gives an absolute minimum for P. We solve to find that zero:

```
FindRoot[P'[t] == 0, {t, 20}]
```

```
{t → 30.173724}
```

The amount of pollutant in the lake is at a minimum when $t \sim 30.174$ days. Let's call that number T.

```
T = t /. %
```

```
30.173724
```

■ c.

By the Fundamental Theorem of Calculus, the amount $P[t]$ of pollutant in the lake satisfies $P[t] - P[0] = \int_0^t P'[\tau]\, d\tau$, or $P[t] = 50 + \int_0^t P'[\tau]\, d\tau$.
Integrating numerically, we find that $P[T]$ is about

```
50 + NIntegrate[P'[t], {t, 0, T}]
```

```
35.104338
```

Because the lake is considered safe when $P \leq 40$, and $P[T] \sim 35.104$, we conclude that the lake is safe when the amount of pollutant in it reaches its minimum.

■ d.

The linearization L at $t = 0$ has equation

```
L[t_] = 50 + P'[0] t
```

```
50 - 2 t
```

Thus, the linearization predicts the arrival of safety when $50 - 2\,t = 40$, or when $t = 5$.

3.

■ a.

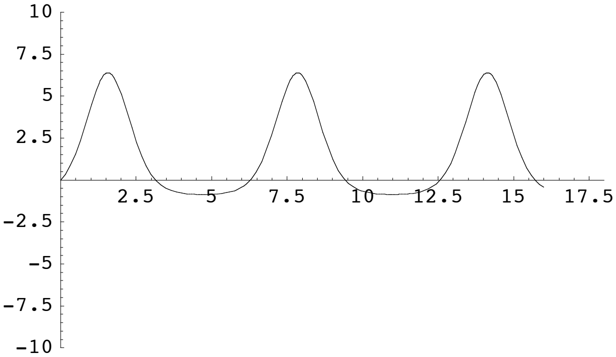
We are given:

```
v[t_] = ⅇ^(2 Sin[t]) - 1
```

```
-1 + ⅇ^(2 Sin[t])
```

The plot should look like this:

```
Plot[v[t], {t, 0, 16}, PlotRange → {{0, 18}, {-10, 10}}]
```



```
- Graphics -
```

b.

The particle moves to the left in those time-intervals where $v[t] < 0$, that is, where $e^{\text{Sin}[t]} < 1$, or, equivalently, where $\text{Sin}[t] < 0$. Thus, motion is to the left when $\pi < t < 2\pi$, When $3\pi < t < 4\pi$, and when $5\pi < t \leq 16$.

c.

We compute distance by integrating speed, which is the absolutevalue of velocity, over the time interval in question. Integrating numerically,
we obtain

```
NIntegrate[v[t], {t, 0, π}] - NIntegrate[v[t], {t, π, 4}]
```

```
10.54247
```

The total distance travelled as t ranges from 0 to 4 is thus about 10.542. The problem gives no units, so we give none.

d.

The particle moves to the right throughout the time interval $0 < t < \pi$. During that interval, it travels about

```
NIntegrate[v[t], {t, 0, π}]
```

```
10.106563
```

It then moves to the left throughout the time interval $\pi < t < 2\pi$. At $t = 2\pi$, we find the particle at about

```
NIntegrate[v[t], {t, 0, 2 π}]
```

```
8.0398716
```

units to the right of the origin. Because it moves leftward throughout this interval, it can't reach the origin any earlier and return to the right. So the particle does not return to the origin during the first interval of length 2π. During the succeeding interval of length 2π, the particle behaves as it did during the first interval of length 2π, but begins at a point 8.03987 units to the right of where it began the first such interval. It therefore doesn't even return as far to the left as the point 8.03987 during the second 2π-interval, let alone to the origin. Similiar reasoning shows that it does not return to the origin on any subsequent interval of the form $(2k\pi,\ 2(k+1)\pi)$, and therefore never returns to the origin.

4.

▪ a.

If $g[x] = 5 + \int_6^x f[t]\,dt$, then $g[6] = 5$, because $\int_6^6 f[t]\,dt = 0$. By the Fundamental Theorem of Calculus, $g'[x] = f[x]$, so $g'[6] = 3$. (We have read the value of $f[6]$ from the graph given in the statement of the problem.) We now see that $g''[x] = f'[x]$, and we have been given that $f'[6] = 0$ (in the equivalent form that the line tangent to the curve at $x = 6$ is horizontal). Hence $g''[6] = 0$.

▪ b.

We know that $g'[x] = f[x]$, and we also know that g is decreasing on any interval where g' is negative. Moreover, if a continuous function is decreasing on an open interval, it is also decreasing on the closure of that interval. From the graph given, we see that $f[x] < 0$ on $[-3, 0)$ and on $(12, 15]$. We conclude that g is decreasing on $[-3, 0]$ and that g is decreasing on $[12, 15]$.

Note: We may not conclude that g is decreasing on $[-3, 0] \cup [12, 15]$. In fact, values of $g[x]$ for x in $[-3, 0]$ will be negative and smaller than those for x in $[12, 15]$--which will be positive.

▪ c.

A function is concave downward on any open interval where its derivative is decreasing. But $g'[x] = f[x]$ is decreasing on the interval $[6, 15]$, and nowhere else. Thus, g is concave downward on $([6,\ 15)$. Whether or not to include the endpoints of this interval is highly dependant upon the definition one uses for "concave downward". There are several distinct definitions in use in different textbooks on the market, and the end-points should not affect scoring.

▪ d.

The required trapezoidal approximation for $\int_{-3}^{15} f[t]\,dt$ is $(f[-3]\ +\ 2f[0] + 2f[3] + 2f[6] + 2f[9] + 2f[12] + f[15]) \cdot \frac{3}{2}$.

Reading function values from the graph, we obtain $(-1 + 0 + 2 + 6 + 2 + 0 + -1) \cdot \frac{3}{2} = 12$.

5.

■ a.

If the line $y = -2$ is tangent to the solution curve $y = f[x]$ of the differential equation $y' = \frac{3-x}{y}$, then at the point of tangency we must have $y' = 0$, or $\frac{3-x}{y} = 0$. This means that the x-coordinate of the point of tangency must be 3. Because the y-coordinate of the point of tangency is necessarily -2, there is then an open interval I centered at $x = 3$ throughout which $f[x] < 0$. We may assume that I does not contain 0. (This is a consequence of the continuity of f, which we infer from its differentiability.) If $x \in I$ and $x < 3$, then $y' = \frac{3-x}{y} < 0$ because $3 - x > 0$ and $y < 0$. If, on the other hand, $x \in I$ and $x > 3$, then $y' = \frac{3-x}{y} > 0$, because $3 - x < 0$ and $y < 0$. Consequently, $x = 3$ is a critical point for f and the derivative of f is negative to the right but positive to the left of this critical point. It follows, by the First Derivative Test, that f has a local minimum at $x = 3$.

Note: This problem can also be solved by finding the solution of the differential equation that passes through the point $(3, -2)$. This solution, which can be found by separation of variables, as below, is $f[x] = -\sqrt{-(x^2 - 6x + 5)}$. Then $f'[x] = \frac{x-3}{\sqrt{-(x^2 - 6x + 5)}}$. Either the First Derivative Test or the Second Derivative Test applies here; the First Derivative Test is more efficient.

■ b.

If $f'[x] = (3 - x)/f[x]$, then $f[x]\, f'[x] = 3 - x$. Consequently, $\int_6^x f[\tau]\, f'[\tau]\, d\tau = \int_6^x (3 - \tau)\, dt$.

Hence $\frac{1}{2} f[x]^2 - \frac{1}{2} f[6]^2 = (3x - \frac{1}{2} x^2) - (18 - 18)$, or $f[x]^2 = 16 + 6x - x^2$. Consequently, $f[x] = -\sqrt{16 + 6x - x^2}$, where we have chosen the minus sign for the square root to reflect that $f[6] = -4$.

6.

■ a.

If $D[t]$ denotes the distance between the two ships at time t, then by the Pythagorean Theorem, $D[t]^2 = x[t]^2 + y[t]^2$. When $x[t] = 4$ km and $y[t] = 3$ km, this gives $D[t] = 5$ km.

■ b.

Differentiating the equation relating $D[t]$, $x[t]$, and $y[t]$ implicitly, we find that $2 D[t]\, D'[t] = 2 x[t]\, x'[t] + 2 y[t]\, y'[t]$, whence $D'[t] = \frac{x[t]\, x'[t] + y[t]\, y'[t]}{D[t]}$. If $x[t] = 4$ km and $y[t] = 3$ km, then, together with $x'[t] = -15$ km/hr and $y'[t] = 10$ km/hr, which were given, and $D[t] = 5$ km, which we found above, this gives $D[t] = \frac{4\cdot(-15)+3\cdot(10)}{5} = -6$ km/hr.

■ c.

Let $\theta[t]$ denote the angle indicated. Then $\tan[\theta[t]] = y[t]/x[t]$, so $\sec^2[\theta[t]]\,\theta'[t] = (y'[t]\,x[t] - x'[t]\,y[t])/x[t]^2$. But when $x[t] = 4$ and $y[t] = 3$, we have $\sec[\theta[t]] = D[t]/x[t] = 5/4$. Hence $\theta'[t] = \left(\frac{4}{5}\right)^2 \left(\frac{10\cdot 4-(-15)\cdot 3}{4^2}\right) = \frac{17}{5}$ radians / hr.