Solutions to the 2003 AP Calculus AB Exam

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Problem 1.

■ a.

The curves intersect where $\sqrt{x} = e^{-3x}$:

$$\mathbf{a = x\ /.\ FindRoot}\left[\sqrt{\mathbf{x}} == e^{-3\,\mathbf{x}},\ \left\{\mathbf{x},\ \frac{1}{2}\right\}\right]\mathbf{[[1]]}$$

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0.2387341
```

The area of the region R is thus

$$\int_{\mathbf{a}}^{1}\left(\sqrt{\mathbf{x}} - e^{-3\,\mathbf{x}}\right)d\mathbf{x}$$

```
0.44262992
```

■ b.

This problem is most easily solved using the method of washers. The required volume is then

$$\pi\int_{\mathbf{a}}^{1}\left((1 - e^{-3\,\mathbf{x}})^2 - \left(1 - \sqrt{\mathbf{x}}\right)^2\right)d\mathbf{x}$$

```
1.4235585
```

It is also possible to use the method of shells. This method leads to

$$2\,\pi\int_{e^{-3}}^{\sqrt{\mathbf{a}}}(1 - \mathbf{y})\left(1 + \frac{1}{3}\,\mathbf{Log[y]}\right)d\mathbf{y} + 2\,\pi\int_{\sqrt{\mathbf{a}}}^{1}(1 - \mathbf{y})\,(1 - \mathbf{y}^2)\,d\mathbf{y}$$

```
1.4235585
```

■ c.

The area of the cross section meeting the x-axis at $x = h$ is

```
A[h_] = 5 (√h - ⅇ^(-3 h))^2 // Expand
```

```
5 ⅇ^(-6 h) - 10 ⅇ^(-3 h) √h + 5 h
```

When we attempt to find $\int_a^1 A[x]\,dx$, we find that the middle term requires numeric integration. Consequently, the required volume is

```
NIntegrate[A[x], {x, a, 1}]
```

```
1.5543544
```

Problem 2

a.

We are given

```
v[t_] = -(t + 1) Sin[t^2/2]
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(-1 - t) Sin[t^2/2]
```

Acceleration is

```
v'[t]
```

```
(-1 - t) t Cos[t^2/2] - Sin[t^2/2]
```

At time $t = 2$, we have an acceleration of

```
v'[2]
```

```
-6 Cos[2] - Sin[2]
```

Speed is $S[t] = |v[t]|$, so $S'[t] = \frac{v[t]}{|v[t]|} v'[t]$. Thus $S'[2]$ is given by

```
v[2.0]/Abs[v[2.0]] v'[2.0]
```

```
-1.5875836
```

$S'[2]$ is negative, so speed is decreasing at time $t = 2$.

■ b.

Changes in direction correspond to local extrema for position--which must occur where the derivative of position, *i.e.*, velocity, changes sign. If $v[t] = 0$, then either $1 + t = 0$ or $\sin\left(\frac{t^2}{2}\right) = 0$. In the interval $0 < t < 3$, $1 + t$ is positive, while $\sin\left(\frac{t^2}{2}\right)$ changes sign only at $\frac{t^2}{2} = \pi$, or $t = \sqrt{2\pi}$. This gives the only time at which a direction change occurs.

■ c.

The total distance (as opposed to total *displacement*) is $\int_0^3 |v[t]| \, dt$, or, integrating numerically,

```
NIntegrate[Abs[v[t]], {t, 0, √2 π}] + NIntegrate[Abs[v[t]], {t, √2 π, 3}]
```

```
4.3338182
```

(*Mathematica*'s numerical integration routines handle the corner at $t = \sqrt{2\pi}$ poorly; thus two integrals instead of one.)

■ d.

Position is given by $x[t] = 1 + \int_0^t v[\tau] \, d\tau$, because we are given $x[0] = 1$. Maximal distance from the origin for $0 \le t \le 3$ must then correspond to either $t = 0$, $t = 3$, or $t = \sqrt{2\pi}$, the latter value being the only critical value for t, as shown in part b. above. We have $x[0] = 1$,

```
x[√2 π] = 1 + NIntegrate[v[τ], {τ, 0, √2 π}]
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-2.2654828
```

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x[3] = 1 + NIntegrate[v[τ], {τ, 0, 3}]
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-1.1971474
```

Maximal distance from the origin therefore occurs when $t = \sqrt{2\pi}$, and is approximately -2.265. (Thanks to Barry Hornstein for pointing out that I dropped the intial value, 1, from both of these calculations in my original version of these solutions.)

Problem 3.

■ a.

An approximate value for $R'[45]$ is $\frac{R[50] - R[40]}{50-40} = \frac{55-40}{10} = \frac{3}{2}$ gallons per minute.

■ b.

If $R[t]$ is increasing fastest at $t = 45$, then $R'[t]$ is maximal when $t = 45$. This means that R' has a critical point at $t = 45$, or that $R''[45] = 0$.

■ c.

The required left Riemann sum is

```
20 (30 - 0) + 30 (40 - 30) + 40 (50 - 40) + 55 (70 - 50) + 65 (90 - 70)

3700
```

Note that $R[t]$ increases throughout the interval $0 \le t \le 90$. Hence $R[t]$ is minimal for each sub-interval that we consider at its left end-point.. This means that $3700 < \int_0^{90} R[t]\, dt$.

■ d.

If $0 \le b \le 90$, then $\int_0^b R[t]\, dt$ is the amount of fuel consumed (in gallons) between $t = 0$ and $t = b$. And $\frac{1}{b} \int_0^b R[t]\, dt$ is the average rate (in gallons/minute) at which fuel has been consumed between $t = 0$ and $t = b$.

Problem 4

■ a.

The graph $f'[x]$, as given, lies above the x-axis only on the interval $[-3, -2)$, so f is increasing on $[-3, -2]$.

■ b.

Inflection points are places where the derivative changes from increasing to decreasing or from decreasing to increasing. For this function, we see from the graph of f' that this occurs when $x = 0$ and when $x = 2$.

■ c.

We have $f'[0] = -2$, so the tangent line at $(0, 3)$ has equation $y = 3 - 2x$.

■ d.

We know by the Fundamental Theorem of Calculus that $f[x] = 3 + \int_0^x f'[t]\,dt$, so $f[-3] = 3 + \int_0^{-3} f'[t]\,dt$. Now $\int_{-3}^0 f[t]\,dt$ is the area of a triangle of base 1 and height 1 minus the area of a triangle of base 2 and height 2, or $\frac{1}{2} - 2 = -\frac{3}{2}$. Thus $f[-3] = 3 + \frac{3}{2} = \frac{9}{2}$. On the other hand, $f[4] = 3 + \int_0^4 f[t]\,dt$, and $\int_0^4 f[t]\,dt$ is the negative of the area that remains when a semi-circle of radius 2 is removed from a rectangle of base 4 and height 2; this is $-(8 - 2\pi)$. Thus $f[4] = 3 - (8 - 2\pi) = 2\pi - 5$.

Problem 5

■ a.

Because $V = \pi r^2 h = 25\pi h$, $\frac{dV}{dt} = 25\pi \frac{dh}{dt}$. But we are given that $\frac{dV}{dt} = -5\pi\sqrt{h}$. Hence $25\pi \frac{dh}{dt} = -5\pi\sqrt{h}$, and, dividing by 25π, we obtain $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$.

■ b.

We have $\frac{1}{\sqrt{h}} \frac{dh}{dt} = -\frac{1}{5}$, so $\int_0^t \frac{1}{\sqrt{h[\tau]}} h'[\tau]\,d\tau = -\int_0^t \frac{d\tau}{5}$.Substituting $u = h[\tau]$ in the integral on the left, we obtain $\int_{17}^{h[t]} \frac{du}{\sqrt{u}} = -\int_0^t \frac{d\tau}{5}$, or $2\sqrt{h[t]} - 2\sqrt{17} = -\frac{1}{5} t$. Thus $h[t] = \left(\sqrt{17} - \frac{1}{10} t\right)^2$.

■ c.

The coffee pot is empty when $h[t] = \left(\sqrt{17} - \frac{1}{10} t\right)^2 = 0$, or when $t = 10\sqrt{17}$ seconds.

Problem 6

■ a.

The function f is continuous at $x = 3$ iff $\lim_{x\to 3} f[x] = f[3]$. Now $f[3] = 2$ is given, and it is clear that $\lim_{x\to 3^-} f[x] = f[3]$--also from what is given. We have $\lim_{x\to 3^+} f[x] = \lim_{x\to 3^+} (5 - x) = 2$. We conclude that f is continuous at $x = 3$.

■ b.

The average value of f over $[0, 5]$ is $\frac{1}{5}\int_0^5 f[x]\,dx = \frac{1}{5}\left(\int_0^3 \sqrt{x+1}\,dx + \int_3^5 (5 - x)\,dx\right) = \frac{1}{5}\left(\frac{2}{3}(x+1)^{3/2}\Big|_0^3 + \left(5x - \frac{x^2}{2}\right)\Big|_3^5\right) = \frac{4}{3}$.

■ c.

If $g'[3]$ is to exist, then g must be continuous at $x = 3$, and, reasoning as above in part a, we find that this imposes the restriction that $2\,k = 3\,m + 2$, or that $k = \frac{3}{2}\,m + 1$. Now suppose that $\lim_{x\to 3^+} g'[x]$ exists, and has the value L. Then, by the Mean Value Theorem, we may write, for positive h, $g[x + h] - g[x] = g'[x^*]\,h$ for some $x^* \in (x,\ x + h)$. Thus, $\lim_{h\to 0^+} \frac{g[x+h] - g[x]}{h} = \lim_{h\to 0^+} g'[x^*] = L$. Arguing in a similar fashion for the left-handed limits, we conclude that $g'[3]$ exists if $\lim_{x\to 3} g'[x]$ exists, and when this is the case, the limit gives $f'[3]$. Now for $x < 3$, we have $g'[x] = \frac{k}{2\sqrt{x+1}}$, which approaches $\frac{k}{4}$ as $x \to 3^-$. For $x > 3$we have $g'[x] = m$, which approaches m as $x \to 3^+$. Thus, the condition for differentiability at $x = 3$ is $m = \frac{k}{4}$. Putting this into the condition for continuity leads to $k = \frac{3}{2}\,\frac{k}{4} + 1$, or $k = \frac{8}{5}$. Subsituting for k in the condition for differentiability now gives $m = \frac{2}{5}$.