Solutions to the 2003 AP Calculus AB Exam (Form B)

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Problem 1.

■ a.

If $f[x] = 4x^2 - x^3$, and $g[x] = 18 - 3x$, then the curves have an intersection in the first quadrant where $x = 3$.

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Solve[4 x^2 - x^3 == 18 - 3 x, x]
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{{x → -2}, {x → 3}, {x → 3}}
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We have $f'[x] = 8x - 3x^2$, so the equation of the line tangent to the curve $y = f[x]$ at $x = 3$ is $y = f[3] + f'[3](x - 3)$,
or $y = (4 \cdot 3^2 - 3^3) + (8 \cdot 3 - 3 \cdot 3^2)(x - 3) = 9 + (24 - 27)(x - 3)$. This is $y = 9 - 3x + 9$, or $y = 18 - 3x$.

■ b.

The curve $y = f[x]$ intersects the x-axis at $x = 4$, and the line crosses the x-axis at $x = 6$. Therefore the area of the region S is

$$\int_3^4 ((18 - 3x) - (4x^2 - x^3))\,dx + \int_4^6 (18 - 3x)\,dx$$

$$\frac{95}{12}$$

■ c.

The curve $y = 4x^2 - x^3$ crosses the x-axis at $x = 0$ and at $x = 4$. Thus, the volume generated when the region R is revolved about the x-axis is

$$\pi \int_0^4 (4x^2 - x^3)^2\,dx$$

$$\frac{16384\pi}{105}$$

(Note: I mistakenly found the area generated by the region S instead of the region R in an earlier version of these solutions. Thanks to Heather Kelly for alerting me to the mistake.)

Problem 2

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H[t_] = 2 + 10/(1 + Log[t + 1])
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$2 + \frac{10}{1 + \text{Log}[1 + t]}$

```
R[t_] = 12 Sin[t^2/47]
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$12\,\text{Sin}\left[\frac{t^2}{47}\right]$

■ a.

The amount pumped into the tank during $0 \le t \le 12$ is given by $\int_0^{12} H[t]\, dt$. Integrating numerically, we obtain

```
NIntegrate[H[t], {t, 0, 12}] gallons
```

70.570859 gallons

■ b.

The rate of change of the volume of oil in the tank at time t is $H[t] - R[t]$. At $t = 6$ this is

```
H[6.0] - R[6.0]
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-2.9241917

The level of oil in the tank is decreasing when $t = 6$ because the volume of oil in the tank has negative rate of change then.

■ c.

We are given that there were 125 gallons of oil in the tank when $t = 0$. Thus, the volume in the tank at time t is, by the Fundamental Theorem of Calculus, $V[t] = 125 + \int_0^t (H[\tau] - R[\tau])\, d\tau$. Integrating numerically with $t = 12$, we obtain

```
(125 + NIntegrate[H[τ] - R[τ], {τ, 0, 12}]) gallons
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122.02571 gallons

■ d.

Here is a plot of the rate at which the volume of oil in the tank changes.

```
Plot[H[t] - R[t], {t, 0, 12}]
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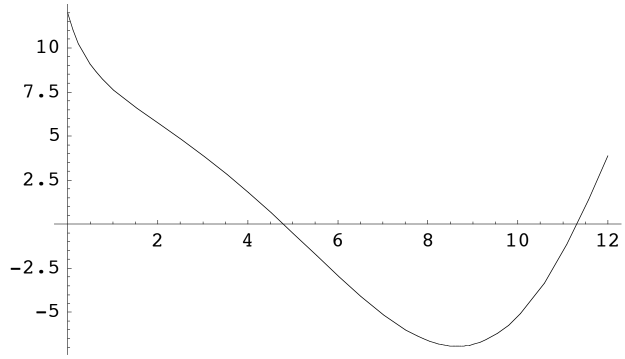

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- Graphics -
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This rate is positive, and the volume increases, from $t = 0$ until about $t = 5$. The rate is negative, and the volume decreases, from about $t = 5$ until a little after $t = 11$. Thereafter, volume increases. This means that there is a minimum somewhere near $t = 11$. Solving numerically, we locate the relevant critical point:

```
t /. FindRoot[H[t] - R[t] == 0, {t, 11.0}][[1]]

11.31847
```

The amount of oil in the tank is minimal when $t = 11.3185$.

Problem 3.

■ a.

Average radius is $\frac{1}{720}\int_0^{360} B[x]\,dx$.

■ b.

The required midpoint Riemann sum is

$$\frac{1}{720}\,(30\,(120 - 0) + 30\,(240 - 120) + 24\,(360 - 240))$$

```
14
```

■ c.

The integral $\pi\int_{125}^{275}\left(\frac{B[x]}{2}\right)^2 dx$ gives the volume, in cubic millimeters, of the segment of the blood vessel that extends from $x = 125$mm to $x = 275$ mm.

■ d.

We have $B[60] = B[180]$. Consequently, by Rolle's Theorem, there is a number $x_1 \in (60, 180)$ where $B'[x_1] = 0$. We also have $B[240] = B[360]$, so, once again by Rolle's Theorem, there is a number $x_2 \in (240, 360)$ where $B'[x_2] = 0$. But then $B'[x_1] = B'[x_2]$, so, yet again by Rolle's Theorem, there must be a number x_3 between x_1 and x_2, such that $B''[x_3] = 0$. Noting that $0 < x_1 < x_2 < 360$, we see that $x_3 \in (0, 360)$.

Problem 4

■ a.

If $v[t] = -1 + e^{1-t}$, then acceleration is $a[t] = -e^{1-t}$. When $t = 3$, we have $a[3] = -e^{-2}$.

■ b.

Speed is $S[t] = |v[t]|$, and so $S'[t] = \frac{v[t]}{|v[t]|} v'[t] = \frac{-1+e^{1-t}}{|-1+e^{1-t}|} (-e^{1-t})$. When $t = 3$, this is $\frac{-1+e^{-2}}{1-e^{-2}} (-e^{-2})$, which is positive. Hence $S[t]$ is increasing when $t = 3$, because its derivative is positive there.

■ c.

The particle changes direction where derivative of position with respect to time, which is velocity, changes sign. This happens only when $-1 + e^{1-t} = 0$, or when $t = 1$.

■ d.

The total distance traveled (as opposed to the total *displacement*) by the particle over the interval $0 \le t \le 3$ is $\int_0^3 |v[t]|\, dt$. Taking into account the sign change at $t = 1$, this is $\int_0^1 (e^{1-t} - 1)\, dt + \int_1^3 (1 - e^{1-t})\, dt = (-e^{1-t} - t)\,|_0^1 + (t + e^{1-t})\,|_1^3$. This reduces to $e - 1 + \frac{1}{e^2}$.

Problem 5

■ a.

$g[3]$ is the area of a rectangle of base 1 and height 2 plus the area of a triangle of base 1 and height 2, or $2 + 1 = 3$. By the Fundamental Theorem of Calculus, $g'[x] = f[x]$, and when $x = 3$, this is $f[3] = 2$. In the interval $(2, 4)$, the FTC tells us that $g'[x] = f[x] = 8 - 2x$. Thus $g''[x] = -2$ in that interval, and $g''[3] = -2$.

■ b.

The average rate of change of g on [0, 3] is $\frac{1}{3}\int_0^3 g'[x]\,dx = \frac{1}{3}\,(g[3] - g[0]) = \frac{1}{3}\,(3 - (-4)) = \frac{7}{3}$. (See part a for the calculation of $g[3]$; for $g[0]$, we have $g[0] = \int_2^0 f[t]\,dt$, which is the negative of the area of a triangle of base 2 and height 4, or −4.)

■ c.

By the Fundamental Theorem of Calculus, $g'[x] = f[x]$. Thus, on the interval (0, 3), $g'[x]$ takes on its average value, $\frac{7}{3}$, at two different points: $x = \frac{7}{6}$, and $x = \frac{17}{6}$.

■ d.

An inflection point occurs where the monotonicity of the derivative changes from increasing to decreasing or vice versa. There are two such points for $g'[x] = f[x]$, (We know $g'[x] = f[x]$ by the Fundamental Theorem of Calculus.) They are $x = 2$ and $x = 5$.

Problem 6

■ a.

If $f'[x] = x\sqrt{f[x]}$, then $f''[x] = \sqrt{f[x]} + \frac{x}{2\sqrt{f[x]}}\,f'[x] = \sqrt{f[x]} + \frac{x}{2\sqrt{f[x]}}\,x\sqrt{f[x]} = \sqrt{f[x]} + \frac{x^2}{2}$. Thus, $f''[3] = \sqrt{25} + \frac{9}{2} = \frac{19}{2}$.

■ b.

We have $\frac{f'[x]}{\sqrt{f[x]}} = x$, so, $\int_3^x \frac{f'[s]}{\sqrt{f[s]}}\,ds = \int_3^x s\,ds = \frac{1}{2}\,(x^2 - 9)$. Substituting $y = f[s]$, $dy = f'[s]\,ds$ in the integral on the left, we transform it to $\int_{25}^{f[x]} \frac{dy}{\sqrt{y}} = 2\sqrt{f[x]} - 10$. Thus $2\sqrt{f[x]} - 10 = \frac{1}{2}\,(x^2 - 9)$, or $\sqrt{f[x]} = \frac{1}{4}\,(x^2 - 9) + 5 = \frac{1}{4}\,(x^2 + 11)$. So $f[x] = \frac{1}{16}\,(x^2 + 11)^2$.