Solutions to the 2004 AP Calculus AB Exam Free Response Questions

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Problem 1.

We are given the traffic flow, i.e., the rate at which cars pass through an intersection, is

```
F[t_] = 82 + 4 Sin[t / 2]
```

$$82 + 4\,\mathrm{Sin}\left[\frac{t}{2}\right]$$

in cars per minute.

■ a.

The function F gives the rate at which cars pass through the intersection, so the total number of cars that pass through the intersection in the 30-minute period $0 \le t \le 30$ is

$$\int_0^{30} F[t]\, dt \;//\; N$$

```
2474.0775
```

To the nearest whole number, this is 2474.

■ b.

```
F'[7] // N
```

```
-1.8729134
```

Traffic flow is decreasing at $t = 7$, because $F'[7] < 0$.

c.

The average value, in **cars per minute** of traffic flow over the interval $10 \le t \le 15$ is

$$\frac{1}{15-10}\int_{10}^{15} F[t]\, dt$$

$$\frac{1}{5}\left(410+8\,\text{Cos}[5]-8\,\text{Cos}\left[\frac{15}{2}\right]\right)$$

Numerically, in **cars per minute:**

```
N[%]

81.899243
```

d.

The average rate of change of the traffic flow, in **cars per minute per minute**, over $10 \le t \le 15$ is $\frac{1}{15-10}\int_{10}^{15} F'[t]\, dt = \frac{1}{5}(F[15] - F[10])$, or

$$\frac{1}{5}(F[15]-F[10])$$

$$\frac{1}{5}\left(-4\,\text{Sin}[5]+4\,\text{Sin}\left[\frac{15}{2}\right]\right)$$

Numerically, in **cars per minute per minute**:

```
N[%]

1.5175394
```

Problem 2

■ a.

```
f[x_] = 2 x (1 - x)
```

```
2 (1 - x) x
```

$g[x\_] = 3\ (x - 1)\ \sqrt{x}$

$3\ (-1 + x)\ \sqrt{x}$

The area of the shaded region is

$$\int_0^1 (f[x] - g[x])\, dx$$

$$\frac{17}{15}$$

Numerically:

```
N[%]
```

```
1.1333333
```

■ b.

The volume of the solid generated by rotating the shaded region about the horizontal line $y = 2$ is

$$\int_0^1 (\pi\ (2 - g[x])^2 - \pi\ (2 - f[x])^2)\, dx$$

$$\frac{103\,\pi}{20}$$

Numerically:

```
N[%]

16.179202
```

■ c.

```
h[x_] = k x (1 - x)

k (1 - x) x
```

The volume of the solid given is $\int_0^1 (h[x] - g[x])^2 \, dx = \int_0^1 \left(3\,(x-1)\sqrt{x} + k\,(x - 1)\,x\right)^2 dx$, so the desired equation is $\int_0^1 \left(3\,(x-1)\sqrt{x} + k\,(x - 1)\,x\right)^2 dx = 15$.

Problem 3.

```
v[t_] = 1 - ArcTan[e^t]

1 - ArcTan[e^t]
```

■ a.

Acceleration at time $t = 2$ is

```
v'[2]
```

$$-\frac{e^2}{1 + e^4}$$

Numerically

```
N[%]

-0.13290111
```

■ b.

Speed is given by $s[t] = |v[t]| = \sqrt{v[t]^2}$, so $s'[t] = \frac{1}{2\sqrt{v[t]^2}} 2\, v[t]\, v'[t] = v'[t] \frac{v[t]}{|v[t]|}$. Hence, $s'[2]$ is

```
v'[2] v[2]/Abs[v[2]]

-(e^2 (1 - ArcTan[e^2]))/((1 + e^4) (-1 + ArcTan[e^2]))
```

Numerically, this is

```
N[%]

0.13290111
```

This is a positive number, so speed is increasing when $t = 2$.

■ c.

Because $\text{Arctan}[e^0] = \frac{\pi}{4} < 1$, $v[0]$ is positive. The function $t \mapsto \text{Arctan}[e^t]$, being a composition of increasing functions, is increasing on the positive half of the t-axis, and $\lim_{t\to\infty} \text{Arctan}[e^t] = \frac{\pi}{2}$. Hence, v is a decreasing function on $[0, \infty)$ and $\lim_{t\to\infty} v[t] < 0$. It follows that $v[T] = 0$ for just one positive T, and because $v[t]$ passes from a region where it is positive to a region where it is negative as t increases through T, that point must give a maximum value for $y[t] = -1 + \int_0^t v[\tau]\, d\tau$. We have $v\{T] = 0$ when $T = \ln \tan 1$. Here is a numeric approximation for T:

```
Log[Tan[1.0]]

0.44302272
```

■ d.

As we have noted above, position is given by $y[t] = -1 + \int_0^t v[\tau]\, d\tau$. Thus

```
y[2] = -1 + NIntegrate[v[τ], {τ, 0, 2}]

-1.3606887
```

Because $2 > T$, where T is the number found in part c, and because, as we saw in part c, $y[t]$ is decreasing for all $t > T$, the particle is moving away from the origin when $t = 2$.

Problem 4

Here we are given: $x^2 + 4y^2 = 7 + 3xy$.

a.

From what is given, we find, by implicit differentiation, that $2x + 8yy' = 3y + 3xy'$, so that $(8y - 3x)y' = 3y - 2x$. Dividing by $(8y - 3x)$, we obtain $\frac{dy}{dx} = \frac{3y-2x}{8y-3x}$, as required.

b. Thanks to Bob Enenstein for pointing out that I misspelled zero as "2" in this one.

If we are to have $y' = 0$, then from part a.) we know that $3y - 2x = 0$, and if $x = 3$ we must have $y = 2$. Substituting $x = 3$ and $y = 2$ into the original equation yields $x^2 + 4y^2 = 3^2 + 4 \cdot 2^2 = 9 + 16 = 25 = 7 + 3 \cdot 3 \cdot 2 = 7 + 3xy$. We conclude that $(3, 2)$ meets our requirements.

c.

From part a.) above, we have $(8y - 3x)y' = 3y - 2x$. Hence, $y''(8y - 3x) + y'(8y' - 3) = 3y' - 2$. At $x = 3$, $y = 2$, we have $y' = 0$. Substitution then gives, at $(3, 2)$, $(8 \cdot 2 - 3 \cdot 3)y'' = -2$. It then follows that at $(3, 2)$ we must have $y'' = -2/7 < 0$. We conclude, by the Second Derivative Test, that the curve has a local maximum at $(3, 2)$.

Problem 5

a.

$g[0] = \frac{1}{2}(2 + 1) \cdot 3 = \frac{9}{2}$; $g'[0] = f[0] = 1$.

■ b.

The function g attains a relative maximum at those points x_0 where $g'[x_0] = f[x_0] = 0$ and $g' = f$ is decreasing in some interval centered at x_0. The only such point is $x = 3$.

■ c.

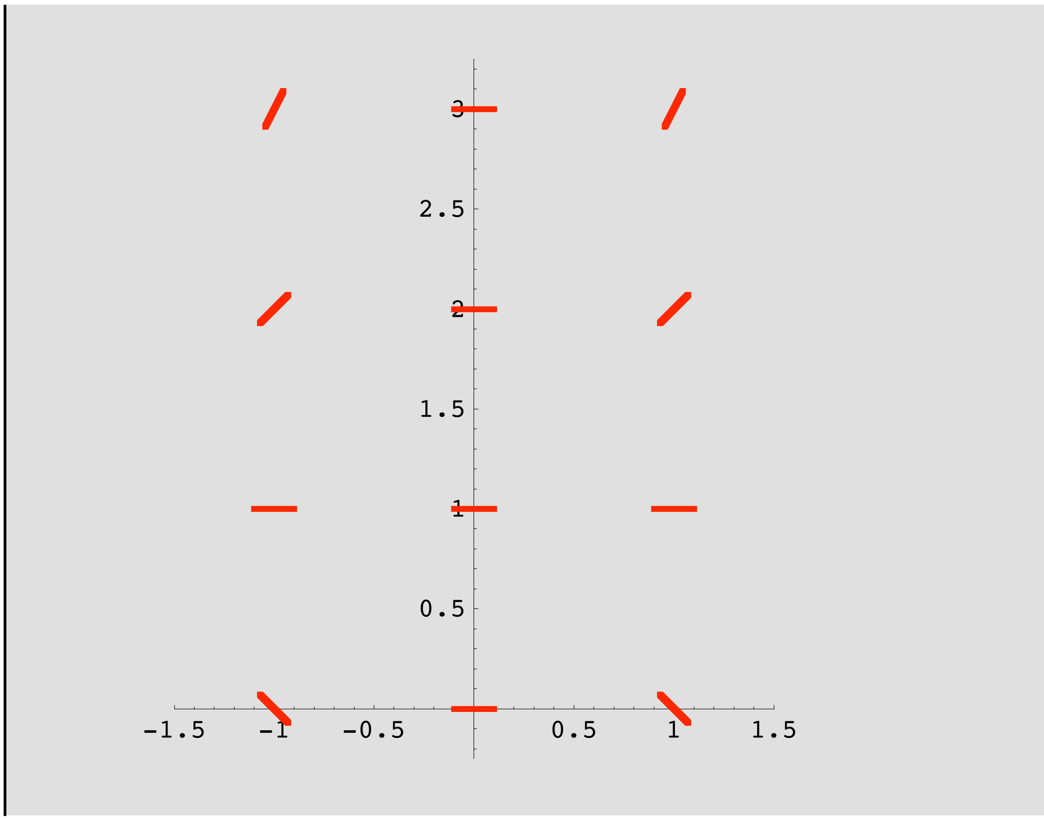
The absolute minimum value of g occurs at either a point where $g' = f$ has a zero or at an end-point of the interval. We have $g'[x] = 0$ at $x = 3$, at $x = 1$, and at $x = -4$. Of these, only $x = -4$ is a possibility, because, by the First Derivative Test, neither of the others is a local minimum--and an absolute minimum interior to the interval must be a local minimum. Now it is geometrically evident that $g[-5] = 0$, $g[-4] = -1$, and $g[4]$ is substantially larger than 0. Consequently, the required absolute minimum vale of g in the interval $[-5, 4]$ is $g[-4] = -1$.

■ d. (Thanks to Dave Slomer for pointing out that I'd skipped this part.)

The graph of g has an inflection point where the graph of $g' = f$ has a relative extremum. Consequently, g has inflection points at $x = -3$, at $x = 1$, and and $x = 2$.

Problem 6

■ a.



■ b.

Because $y' = x^2(y - 1)$, slope is positive only when both x^2 and $(y - 1)$ are positive. Thus, slope is positive precisely where both $x \neq 0$ and $y > 1$.

■ c.

We must have $\int_0^x \frac{y'[\eta]\, d\eta}{y[\eta]-1} = \int_0^x \eta^2\, d\eta$, or $\ln(y[x]-1) - \ln(y[0] - 1) = \frac{1}{3}x^3$. (We need not worry about absolute values inside the logarithms; the nature of the integrand on the left side of our first equation, together with the fact that $y[0] = 3$, makes it clear that we must have $y > 1$.) But $y[0] = 3$, so $\ln(y[x]-1) = \frac{1}{3}x^3 + \ln 2$. Hence $y[x] = 1 + \exp(\frac{1}{3}x^3 + \ln 2) = 1 + 2\exp(\frac{1}{3}x^3)$.