Solutions to the
2004 AP Calculus AB Exam (Form B)
Free Response Questions

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Problem 1.

■ a.

The curve intersects the x-axis at $x = 1$, so the desired area is

$$\int_{1}^{10} \sqrt{x - 1}\, dx$$

$$18$$

■ b.

The volume generated when the region is revolved about the horizontal line $y = 3$ is

$$\pi \int_{1}^{10} \left(9 - \left(3 - \sqrt{x - 1}\right)^{2}\right) dx$$

$$\frac{135\,\pi}{2}$$

Numerically, this is

```
N[%]
```

```
212.0575
```

■ c.

Solving the equation $y = \sqrt{x-1}$ for x in terms of y gives $x = y^2 + 1$. Hence, the volume generated by revolving the region about the vertical line $x = 10$ is

$$\pi \int_0^3 (10 - (y^2 + 1))^2 \, dy$$

$$\frac{648\,\pi}{5}$$

Numerically:

```
N[%]

407.15041
```

Problem 2

■ a.

Because $R[t] = 5\sqrt{t}\cos(\frac{t}{5})$ is the rate of change of the number of mosquitoes on the island, and we have

$$\text{R[t\_]} = 5\sqrt{t}\ \text{Cos}\left[\frac{t}{5}\right]$$

$$5\sqrt{t}\ \text{Cos}\left[\frac{t}{5}\right]$$

```
R[6.0]

4.437958
```

we see that $R[6] > 0$. Because R is continuous, it follows that the number of mosquitoes is increasing in some interval centered at $t = 6$.

b.

```
R'[t]
```

$$\frac{5\,\text{Cos}\left[\frac{t}{5}\right]}{2\sqrt{t}} - \sqrt{t}\,\text{Sin}\left[\frac{t}{5}\right]$$

```
R'[6.0]

-1.9131903
```

$R'[6] < 0$, and R' is continuous near $t = 6$. It follows that $R[t]$ is decreasing near $t = 6$. Thus, the number of mosquitos is increasing at a decreasing rate near $t = 6$.

c.

The number $M[t]$ of mosquitoes at time t is given by $M[t] = M[0] + \int_0^t R[\tau]\,d\tau$, where $M[0] = 1000$. Thus, integrating numerically, we obtain:

```
M[t_] := 1000 + NIntegrate[R[τ], {τ, 0, t}]
```

```
M[31]

964.33519
```

To the nearest whole number, this is 964.

d.

The maximum number of mosquitoes for the period $0 \le t \le 31$ will occur when $t = 0$, or when $t = 31$, or when $R[t] = 0$. The latter condition obtains when $t = 0$, when $t = \frac{5\pi}{2}$, and when $t = \frac{15\pi}{2}$. We have

```
M[0]

1000.
```

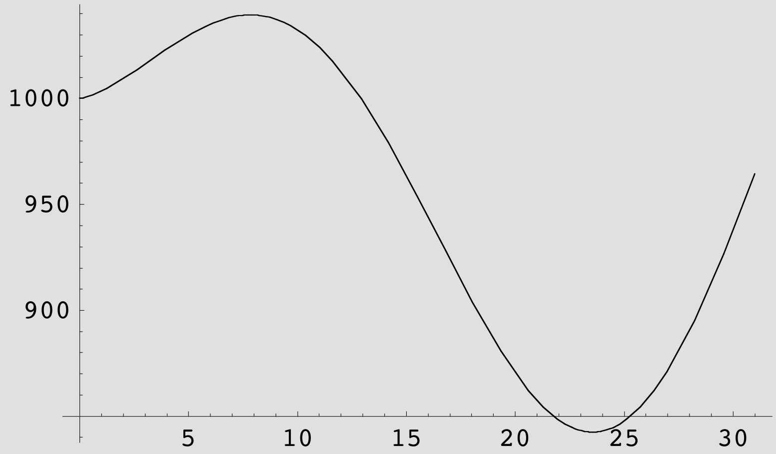
```
M[5 π/2]

1039.3569
```

```
M[15 π/2]

842.40475
```

```
M[31]

964.33519
```

The mosquito population thus peaks at about 1039 when $t = 5\pi/2$. A graph confirms this:

```
Plot[M[t], {t, 0, 31}]
```



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- Graphics -
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Problem 3.

We are given

```
Inner[Set, Map[v, {0, 5, 10, 15, 20, 25, 30, 35, 40}],
 {7.0, 9.2, 9.5, 7.0, 4.5, 2.4, 2.4, 4.3, 7.3}, List]

{7., 9.2, 9.5, 7., 4.5, 2.4, 2.4, 4.3, 7.3}
```

(This arcane syntax assigns the correct values to v[0], v[5], etc.)

a.

The Mid-Point Rule with four subintervals of equal length gives for $\int_0^{40} v[t]\, dt$ the approximate value

$$\sum_{k=1}^{4} 10\, v[5 + 10\, (k - 1)]$$

```
229.
```

$v[t]$ is given in miles per minute, so the integral gives miles travelled during the time interval over which the integral is taken.

b.

By Rolle's Theorem, acceleration--which is $v'[t]$--must be zero at least once in the interval $0 \le t \le 15$, because $v[0] = v[15]$. Similarly, $v'[t]$ must be zero at least once in the interval $25 \le t \le 30$, because $v[25] = v[30]$. Thus, acceleration must vanish at least twice in the interval $0 \le t \le 40$.

c.

$$f[t\_] = 6 + \mathrm{Cos}\left[\frac{t}{10}\right] + 3\, \mathrm{Sin}\left[7\, \frac{t}{40}\right]$$

$$6 + \mathrm{Cos}\left[\frac{t}{10}\right] + 3\, \mathrm{Sin}\left[\frac{7\, t}{40}\right]$$

If the function f models velocity, then acceleration is f'. Thus

```
f'[t]
```

$$\frac{21}{40}\,\mathrm{Cos}\left[\frac{7\,t}{40}\right]-\frac{1}{10}\,\mathrm{Sin}\left[\frac{t}{10}\right]$$

```
% /. t → 23.0
```

```
-0.40769419
```

At time $t = 23$, acceleration is -0.408 miles / minute^2.

■ d.

Average velocity over $0 \le t \le 40$ is $\frac{1}{40}\int_0^{40} f[t]\,dt$:

$$\frac{1}{40}\int_0^{40} \mathbf{f[t]}\,dt$$

$$\frac{1}{40}\left(240+\frac{240}{7}\,\mathrm{Sin}\left[\frac{7}{2}\right]^2+10\,\mathrm{Sin}[4]\right)$$

```
N[%]
```

```
5.9162698
```

Problem 4

■ a.

Inflection points are to be found where f' has relative extrema. Consequently, the function f whose derivative is pictured has inflection points at $x = 1$ and at $x = 3$.

b. Thanks to Tammy Brown, Becky Myers, and Sue Wall, who all pointed out that I'd blown this one completely.

The function f is decreasing on the interval $[-1,\ 4]$ and increasing on the interval $[4,\ 5]$ because f' is non-positive on the first of these intervals, and non-negative on the second. Consequently, f takes on its absolute minimum value for the interval $[-1, 5]$ at $x = 4$.

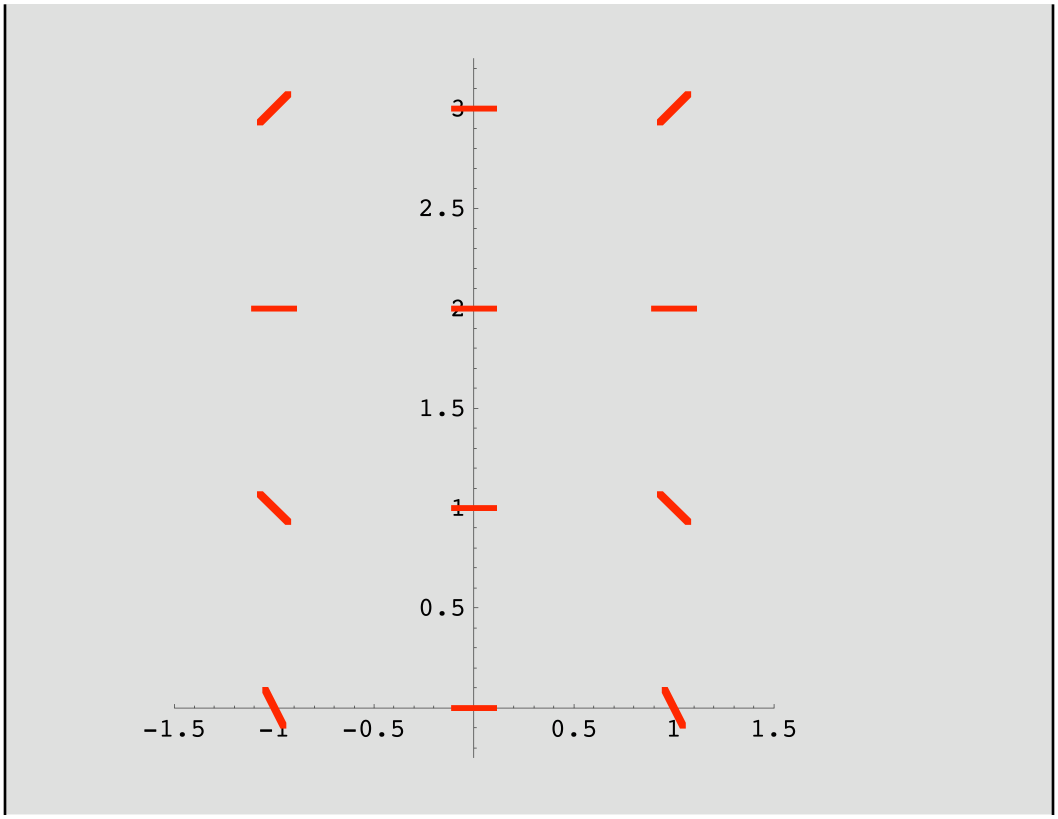
The function f has its absolute maximum at one of the points where $x = -1$ or $x = 5$. (There can be no absolute maximum for f at any point interior to $(-1,\ 5)$ because f' has in that interval no zeroes at which it undergoes a sign change from positive to negative as x increases.) The area bounded by f and the x-axis on the interval $[-1, 4]$ is clearly larger than the area bounded by f and the x-axis on the interval $[4, 5]$, so $\int_4^{-1} f'[t]\,dt = f[-1] - f[4] > f[5] - f[4] = \int_4^5 f'[t]\,dt$. This means that $f[-1] > f[5]$, so that the absolute maximum value taken on by f in the interval $[-1, 5]$ is $f[-1]$.

c.

We are given $g[x] = x\,f[x]$, so $g'[2] = f[2] + 2 \cdot f'[2] = 6 + 2 \cdot (-1) = 4$. Because $g[2] = 2\,f[2] = 12$, this means that an equation for the line tangent to the graph of g at $x = 2$ is $y = 12 + 4\,(x - 2)$.

Problem 5

a.



b. Thanks to Laura Farrell, who pointed out that I identified the region where slope is positive, instead of the region where slope is negative.

Because slope at (x, y) is given by $y' = x^4(y - 2)$, slope is negative only where x^4 is positive and $(y - 2)$ is negative. So slope is negative in $\{(x, y) : x \neq 0 \text{ and } y < 2\}$.

c.

$\frac{1}{f[t]-2} f'[t] = x^4$, with $f[0] = 0$, so $\int_0^x \frac{f'[\xi]}{f[\xi]-2}\, d\xi = \int_0^x \xi^4\, d\xi$. Because $f[0] = 0$, the nature of the first integrand assures us that we are interested only in values of f which are less than 2. Thus, $\ln(2 - f[x]) - \ln 2 = \frac{1}{5} x^5$, or $f[x] = 2 - 2\exp[\frac{1}{5} x^5]$.

Problem 6

a.

We have $\int_0^1 x^n \, dx = \left.\frac{x^{n+1}}{n+1}\right|_0^1 = \frac{1}{n+1}$.

b.

If $y = x^n$, then $y' = n\,x^{n-1}$, so that the equation of the line ℓ is $y = 1 + n(x - 1)$. This line meets the x-axis when $x = 1 - \frac{1}{n}$, so that the base of the triangle T has length $\frac{1}{n}$. Because the altitude of the triangle T is 1, the area of T is $\frac{1}{2\,n}$.

c.

From what we have seen in parts a.) and b.), the area $A[n]$ of the region S is $A[n] = \frac{1}{n+1} - \frac{1}{2\,n} = \frac{n-1}{2\,n(n+1)}$. Then $A'[n] = -\frac{n^2 - 2\,n - 1}{2\,n^2 (n+1)^2}$, which is zero for $n > 1$ only when $n = 1 + \sqrt{2}$ (by the Quadratic Formula). Noting that $A'[n] > 0$ for $1 \leq n < 1 + \sqrt{2}$, while $A'[n] < 0$ for $1 + \sqrt{2} < n$, we conclude that the maximal area occurs when $n = 1 + \sqrt{2}$.