Solutions to the 2005 AP Calculus AB Exam Free Response Questions

Louis A. Talman
Department of Mathematical & Computer Sciences
Metropolitan State College of Denver

Problem 1.

a)

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FindRoot[1/4 + Sin[π x] == 4^-x, {x, 0}]
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{x → 0.17821805}

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a = x /. %
```

0.17821805

So the required area is

$$\int_0^a \left(4^{-x} - \frac{1}{4} - \mathrm{Sin}[\pi\, x]\right) dx$$

0.064753062

b)

$$\int_a^1 \left(\frac{1}{4} + \text{Sin}[\pi\, x] - 4^{-x}\right) dx$$

0.41036219

c)

$$\pi \int_a^1 \left(\left(\frac{1}{4} + \text{Sin}[\pi\, x] + 1\right)^2 - (4^{-x} + 1)^2\right) dx$$

4.5587633

Problem 2.

a)

```
R[t_] = 2 + 5 Sin[4 π t / 25]
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$$2 + 5\,\text{Sin}\left[\frac{4\,\pi\,t}{25}\right]$$

$$S[t\_] = \frac{15\,t}{1 + 3\,t}$$

$$\frac{15\,t}{1 + 3\,t}$$

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∫₀⁶ R[t] dt
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$$12 + \frac{125}{4\pi} - \frac{125\,\mathrm{Cos}\left[\frac{24\pi}{25}\right]}{4\pi}$$

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N[%]
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```
31.815931
```

The tide will remove 31.816 cubic yards of sand during this 6-hour period.

b)

$Y(t) = 2500 - \int_0^t (2 + 5\sin[4\pi\tau/25])\,d\tau + \int_0^t \frac{15\tau}{1+3\tau}\,d\tau.$

c)

When $t = 4$ the total amount of sand is changing at the rate

```
S[4] - R[4]
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$$\frac{34}{13} - 5\,\mathrm{Sin}\left[\frac{16\pi}{25}\right]$$

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N[%]
```

```
-1.9087506
```

d)

We examine a plot of the rate at which sand accumulates on the beach:

```
Plot[S[t] - R[t], {t, 0, 6}]
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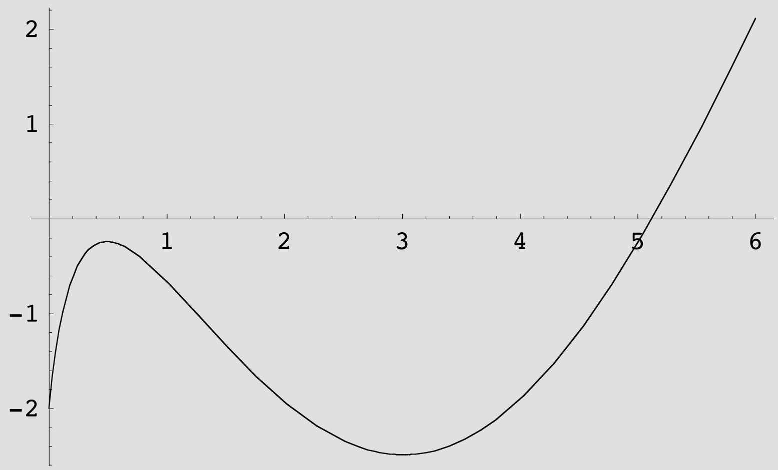


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- Graphics -
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As the plot shows, the rate of accumulation is negative when $t < t_0$ and positive when $t > t_0$, for a certain value t_0 which is approximately 5. We solve for t_0, which must be the time when the amount of sand on the beach is minimal:

```
FindRoot[S[t] - R[t] == 0, {t, 5}]

{t → 5.1178653}
```

The amount of sand on the beach is minimal at about $t = 5.118$ hours. The minimal amount is about

$$2500 - \int_0^{5.117865253093577`} (2 + 5\,\mathrm{Sin}[4\,\pi\,\tau / 25])\, d\tau + \int_0^{5.117865253093577`} \frac{15\,\tau}{1 + 3\,\tau}\, d\tau$$

```
2492.3695
```

Problem 3.

a)

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55 - 62
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 8 - 6
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$-\frac{7}{2}$

Thus $T'(7)$ is approximately $\frac{T(8)-T(6)}{8-6}$ degrees C per centimeter, or -7/2 degrees C. per centimeter.

b)

Average temperature of the wire is $\frac{1}{8}\int_0^8 T(x)\,dx$. This is

$$\frac{1}{8}\left(\frac{100+93}{2}+\frac{93+70}{2}(5-1)+\frac{70+62}{2}+\frac{62+55}{2}(8-6)\right)$$

$$\frac{1211}{16}$$

Answer: 1211 / 16 degrees Celsius.

c)

We are given that T is twice differentiable (although we are not told where). This means that T' is continuous--on the interval [0,8], we hope, so that the problem is meaningful. By the Fundamental Theorem of Calculus, $\int_0^8 T'(x)\,dx = T(8) - T(0) = -45$ degrees Celsius. The integrand $T'(x)$ is the (instantaneous) rate at which $T(x)$ changes per unit length at each point of the interval [0, 8], and the integral gives net temperature change over the interval [0, 8].

d)

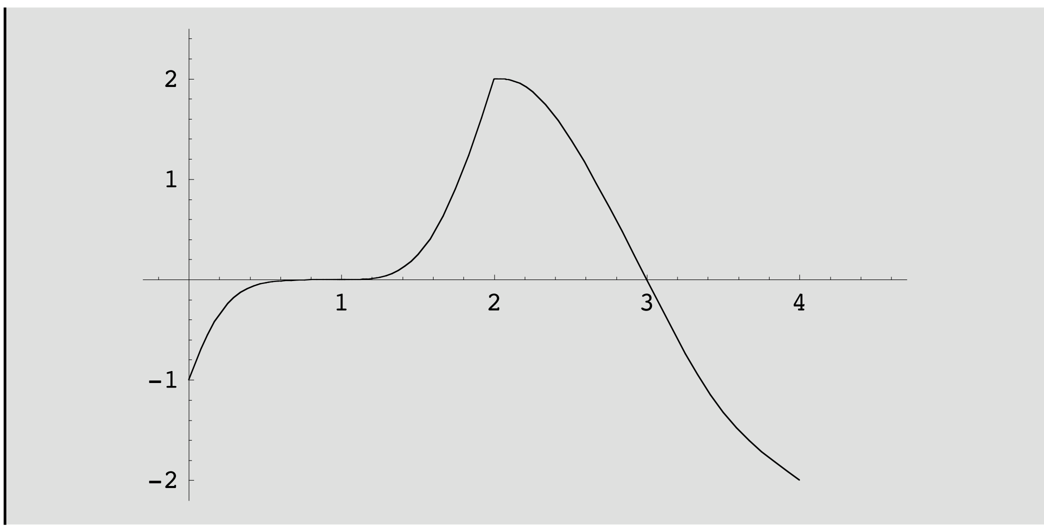
By the Mean Value Theorem, there is a point $\xi \in [1, 5]$ where $T'(\xi) = [T(5) - T(1)]/(5-1) = -23/4$. By the Mean Value Theorem again, there is a point $\eta \in [5, 6]$ where $T'(\eta) = T(6) - T(5) = -8$. Note that, necessarily, $0 < \xi < \eta < 8$. A third application of the Mean Value Theorem---this time to T'---assures us that there is a point $\zeta \in [\xi, \eta] \subseteq (0, 8)$ such that $T''(\zeta) = [T'(\eta) - T'(\xi)]/(\eta - \xi) = (-8 + 23/4)/(\eta - \xi) = -9/[4(\eta - \xi)] < 0$. Consequently, these data are not consistent with the assertion that $T''(x) > 0$ for every x in (0, 8).

Problem 4.

a)

The derivative of f must be zero or non-existent at any point of (0, 4) where f has a relative extremum. Thus, $x = 1$ and $x = 2$ are the only values we need to consider. We find that $f'(x) > 0$ for all $x \neq 1$ in the interval (0, 2). Consequently, f increases throughout that interval and can't have a relative extremum at $x = 1$. At $x = 2$, we find that $f(2)$ is meaningful, and that $f'(x)$ is positive on the interval (1, 2) but negative on the interval (2, 3). Consequently, f is increasing on (1, 2) but decreasing on (2, 3). This means that f has a relative maximum at $x = 2$.

b)



c)

If $g(x) = \int_1^x f(t)\,dt$ on (0, 4), then $g'(x) = f(x)$ by the Fundamental Theorem of calculus. Thus, g can have relative extrema only at points where $f(x) = 0$. At $x = 1$, $f(x)$ undergoes a sign change from negative to positive, and so $g(x)$ passes from a region where it is decreasing into a region where it is increasing as x passes through 1. Consequently, g has a relative minimum at $x = 1$. Similar reasoning shows that g has a relative maximum at $x = 3$.

d)

The points of inflection for g are to be found where g' has relative extrema. But g' is f, and, according to part a) of this problem, f has a relative extremum only at $x = 1$. We conclude that g has just one inflection point, at $x = 1$.

Problem 5.

■ a)

$\int_0^{24} v(t)\,dt = \frac{1}{2}\cdot(4-0)\cdot 20 + (16-4)\cdot 20 + \frac{1}{2}\cdot(24-16)\cdot 20 = 40 + 240 + 80 = 360$ meters. $\int_0^{24} v(t)\,dt$ gives the distance, in meters, traveled by the car during the time period $0 \le t \le 24$.

■ b)

$v'(4) = \lim_{h\to 0}[v(4+h)-v(4)]/h$ is the definition of $v'(4)$. We note that for $0 < t \le 4$, we have $v(t) = 5t$, so that $v(4+h) - v(4) = 5(4+h) - 5\cdot 4 = 5h$ when $h < 0$ but $|h|$ is small. Thus $[v(4+h)-v(4)]/h = 5$ for such values of h. But when $4 \le t \le 16$, we have $v(t) = 20$, so that $[v(4+h)-v(4)] = 20 - 20 = 0$ when $h > 0$ and $|h|$ is small. For such h we therefore have $[v(4+h)-v(4)]/h = 0$. Consequently, $\lim_{h\to 0+}[v(4+h)-v(4)]/h = 0$, but $\lim_{h\to 0-}[v(4+h)-v(4)]/h = 5$. The two one-sided limits have different values, so the two-sided limit does not exist. this means that $v'(4)$ does not exist.

Using $v(t) = 60 - \frac{5}{2}t$ when $16 \le t < 24$, we find that $\lim_{h\to 0}[v(20+h)-v(20)]/h = \lim_{h\to 0}[(60 - \frac{5}{2}t - \frac{5}{2}h) - (60 - \frac{5}{2}t)]/h$. This is $\lim_{h\to 0}(-5h)/(2h) = -\frac{5}{2}$. It follows that $v'(20) = -\frac{5}{2}$.

■ c)

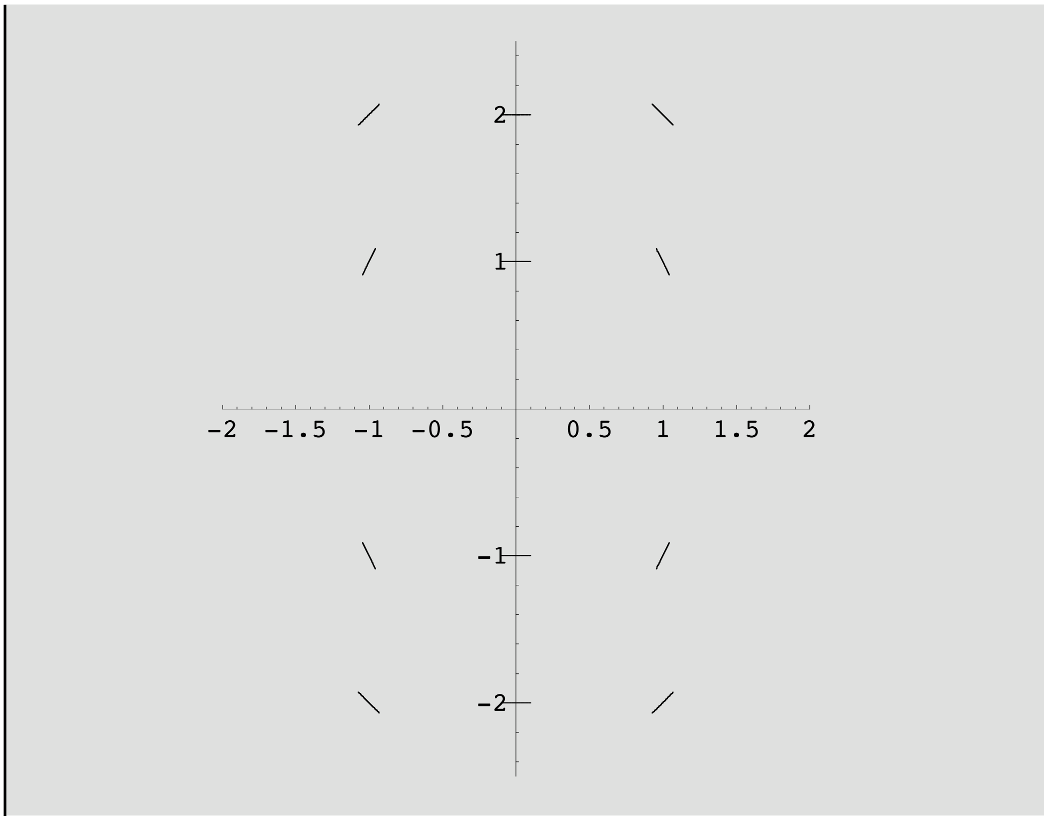
Acceleration is given by $a(t) = 5$ when $0 < t < 4$, by $a(t) = 0$ when $4 < t < 16$, and by $a(t) = -\frac{5}{2}$ when $16 < t < 24$.

■ d)

The average rate of change of v over $8 \le t \le 20$ is $[v(20)-v(8)]/(20-8) = (10-20)/(20-8) = -5/6$. The hypotheses of the Mean Value Theorem require that $v'(t)$ exist at every point of the interior of an interval on which we wish to apply the theorem. The Mean Value Theorem is therefore inapplicable to v on $[8, 20]$ because $v'(16)$ does not exist.

Problem 6.

a)



b)

At $(1, -1)$, we have $y' = -2(1)/(-1) = 2$, so the equation of the line tangent to the solution for which $y(1) = -1$ is $y = -1 + 2(x - 1)$, or $y = 2x - 3$. This gives an approximate value for $y(1.1)$ of $y = 2 \cdot (1.1) - 3 = -0.8$.

c)

If $y'(x) = -2x/y(x)$, then $y(x)\,y'(x) = -2x$. Thus $\int_1^x y(t)\,y'(t)\,dt = -2\int_1^x t\,dt$, or $\frac{1}{2}[y(x)]^2 - \frac{1}{2}[y(1)]^2 = -x^2 + 1^2$. Replacing $y(1)$ with -1 and simplifying, we obtain $[y(x)]^2 = 3 - 2x^2$. This leads to $y(x) = \pm\sqrt{3 - 2x^2}$. We resolve the ambiguity of sign by noting that we must have $y(1) = -1$, and this gives our solution: $y(x) = -\sqrt{3 - 2x^2}$.