Solutions to the
2005 AP Calculus AB (Form B) Exam
Free Response Questions

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Problem 1.

```
f[x_] = 1 + Sin[2 x]

1 + Sin[2 x]
```

```
g[x_] = e^(x/2)

e^(x/2)
```

```
FindRoot[f[x] == g[x], {x, 1}]

{x → 1.1356925}
```

```
a = x /. %

1.1356925
```

a)

$$\int_0^a (f[x] - g[x])\, dx$$

0.42910092

b)

$$\pi \int_0^a ((f[x])^2 - (g[x])^2)\, dx$$

4.2665465

c)

$$\frac{\pi}{2} \int_0^a \left(\frac{1}{2} (f[x] - g[x])\right)^2 dx$$

0.077657313

Problem 2.

$$W[t\_] = 95 \sqrt{t}\, (Sin[t/6])^2$$

$$95 \sqrt{t}\, Sin\left[\frac{t}{6}\right]^2$$

```
R[t_] = 275 (Sin[t / 3])^2
```

$275\ \mathrm{Sin}\left[\frac{t}{3}\right]^2$

a)

```
W[15] - R[15]
```

$95\sqrt{15}\ \mathrm{Sin}\left[\frac{5}{2}\right]^2 - 275\ \mathrm{Sin}[5]^2$

```
N[%]
```

```
-121.09003
```

When t = 15, The difference is negative, so water is being removed from the tank at a higher rate than it is being pumped in, so the amount of water in the tank at that time is decreasing.

b)

The amount of water in the tank at time t is $A[t] = 1200 + \int_0^t (W[\tau] - R[\tau])\, d\tau$. Integrating numerically, we find that $A[18]$ is

```
1200 + NIntegrate[(W[τ] - R[τ]), {τ, 0, 18}]
```

```
1309.7882
```

To the nearest whole number, this is 1310 gallons.

c)

We have $A'[t] = W(t) - R(t)$. We therefore seek the zeros of $W[t] - R[t]$. From the graph,

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Plot[W[t] - R[t], {t, 0, 18}]
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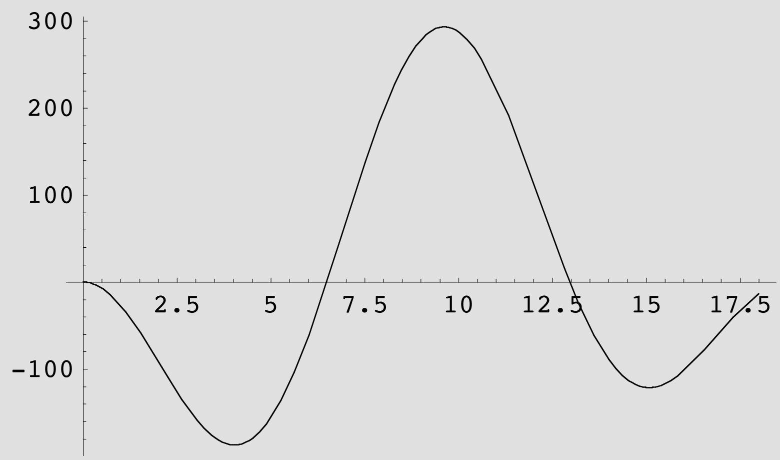



```
- Graphics -
```

we see that these zeros are near $t = 0$, $t = 6.5$, and $t = 13.0$. The first of these is at $t = 0$. We solve numerically for the second and the third.

```
FindRoot[W[t] - R[t] == 0, {t, 6.5}]

{t → 6.4948402}
```

```
r2 = t /. %

6.4948402
```

```
FindRoot[W[t] - R[t] == 0, {t, 13}]

{t → 12.974825}
```

```
r3 = t /. %

12.974825
```

We calculate A for each of these and for $t = 18$. First we note that $A[0] = 1200$.

```
A[t_] := 1200 + NIntegrate[(W[τ] - R[τ]) , {τ, 0, t}]
```

```
A[r2]

525.24215
```

```
A[r3]

1697.4412
```

```
A[18]

1309.7882
```

The amount of water in the tank is minimal when $t \sim 6.495$.

d)

With A and R defined as above, we must solve $A(18) - \int_{18}^{k} R(\tau)\, d\tau = 0$ for k.

Problem 3.

```
v[t_] = Log[t^2 - 3 t + 3]

Log[3 - 3 t + t^2]
```

a)

Acceleration is

```
v'[t]
```

$$\frac{-3 + 2\,t}{3 - 3\,t + t^2}$$

At $t = 4$, this is

```
v'[4]
```

$$\frac{5}{7}$$

■ b)

The particle changes direction where v changes sign. This can occur only where $v(t) = 0$, or where $t^2 - 3t + 3 = 1$. Thus, $t^2 - 3t + 2 = 0$, or $(t-1)(t-2) = 0$. $v(t) = 0$, therefore, when $t = 1$ and when $t = 2$. Now $t^2 - 3t + 2$ has a graph which is a parabola opening upward. Consequently, $v(t) > 0$ when $t < 1$ and when $t > 2$. $v(t) < 0$ when $1 < t < 2$. This means that the particle changes direction at $t = 1$ and at $t = 2$.

■ c)

The position of the particle at time t is $x(t) = 8 + \int_0^t v(\tau)\,d\tau$. When $t = 2$, this is

$$8 + \int_0^2 v[t]\,dt$$

$$4 + \frac{\sqrt{3}\,\pi}{2} + \frac{\text{Log}[27]}{2}$$

Numerically, this is

```
N[%]
```

```
8.3686175
```

■ d)

Average speed on the interval [0, 2] is $\frac{1}{2}\int_0^2 |v(\tau)|\,d\tau$. Using what we have learned about the sign of v in part b) above, we find that the required average speed is;

```
1/2 (∫_0^1 v[τ] ⅆτ - ∫_1^2 v[τ] ⅆτ)
```

$$\frac{1}{2}\left(-\frac{\pi}{2\sqrt{3}}+\frac{3\,\text{Log}[3]}{2}\right)$$

Numerically, this is

```
N[%]

0.37050938
```

Problem 4.

a)

$g(-1) = \int_{-4}^{x} f(t)\,dt$ is the negative of the area of the trapezoid defined by the x-axis, the vertical lines $x=-4$ and $x=-1$ and the line segment joining the points $(-4,-3)$ and $(-1,-2)$. Thus, $g(-1) = \frac{-1}{2}(3+2)\cdot 3 = \frac{-15}{2}$. By the Fundamental Theorem of Calculus, $g'(x) = f(x)$, so $f'(-1) = -2$. Because $g'(x) = f(x)$, it follows that $g''(x) = f'(x)$, iff the latter exists. Because of the corner in the graph of $f(x)$ at the point corresponding to $x=-1$, $f'(-1)$ does not exist. (In fact, $f_{-}'(-1) = 1/3$, while $f_{+}'(-1) = 2$.) Thus, $g''(-1)$ does not exist.

b)

The inflection points of g occur at the values of x for which $g' = f$ has relative extrema. But f has just one relative extremum in the interval $(-4, 3)$, at $x = 1$--as is evident from the graph. Thus the only relative extremum for g is to be found at $x = 1$.

c)

If $h(x) = \int_{x}^{3} f(t)\,dt = -\int_{3}^{x} f(t)\,dt$, then the zeros of h are to be found at those values of x for which the graph of f has just as much area above the x-axis as below in the interval $[x, 3]$. These values are evidently $x = -1$ an $x = 1$. And, of course, we shouldn't forget the trivial solution $x = 3$.

■ d)

With h given as in part c), above, we have by the Fundamental Theorem of Calculus $h'(x) = -f(x)$. Therefore, h is decreasing on the closures of those intervals for which $-f(x) < 0$, or, equivalently, where $f'(x) > 0$. From the graph, it is evident that h is decreasing on [0, 2].

Problem 5.

■ a)

By implicit differentiation, we have $2\,y\,y' = y + x\,y'$, so that $(2\,y - x)\,y' = y$, or $y' = y/(2\,y - x)$, provided that $2\,y - x \neq 0$. But $2\,y - x = 0$ implies that $x = 2\,y$, and, substituting this in the original equation, we find that $y^2 = 2 + (2\,y)\,y$, or $y^2 + 2 = 0$. This is not possible for real y, so we conclude that $y' = y/(2\,y - x)$ whenever (x, y) lies on the curve $y^2 = 2 + x\,y$.

■ b)

If $y' = 1/2$, then $1/2 = y/(2\,y - x)$, or $2\,y - x = 2\,y$. Thus $x = 0$. Now if (x, y) lies on the curve $y^2 = 2 + x\,y$ and $x = 0$, then $y^2 = 2$. The required points are thus $\left(0, \sqrt{2}\right)$ and $\left(0, -\sqrt{2}\right)$.

■ c)

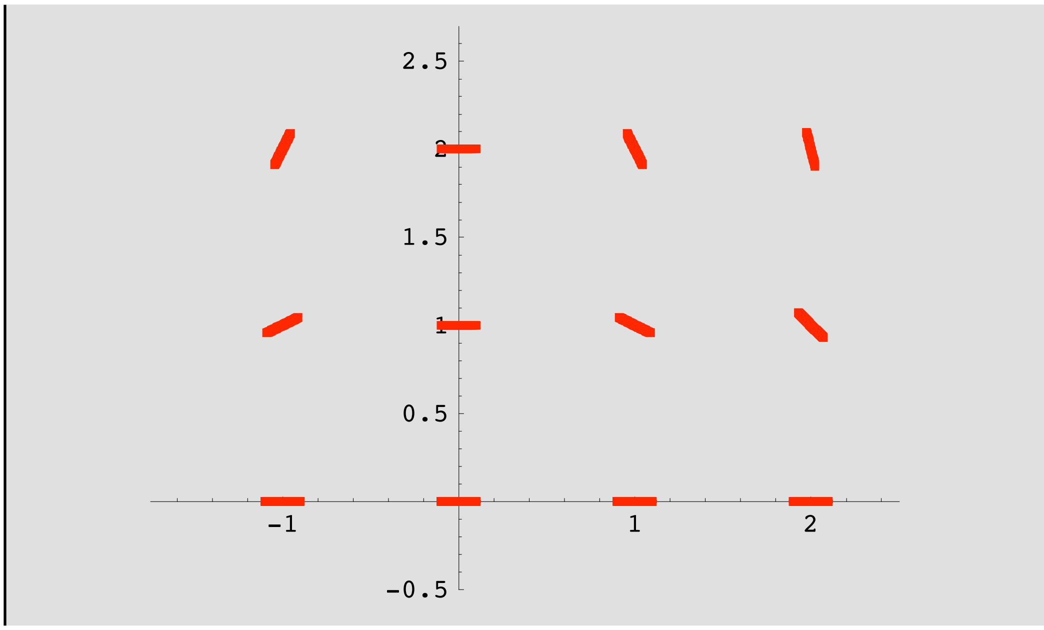
If the tangent line to $y^2 = 2 + x\,y$ is horizontal at a point (x_0, y_0), then we must have $0 = y'(x_0) = y_0/(2\,y_0 - x_0)$, and this implies that $y_0 = 0$. But then $0 = y_0^2 = 2 + x_0\,y_0 = 2 + 0 = 2$, or $0 = 2$. The contradiction shows that there can be no point on the curve $y^2 = 2 + x\,y$ where the tangent line is horizontal.

■ d)

We differentiate the equation for the curve implicitly again, but this time we treat x and y both as functions of t and the prime means differentiation with respect to t. This gives $2\,y\,y' = x'\,y + x\,y'$. Putting $y = 3$, $y' = 6$ in both the original equation and the derived equation leads to the system $9 = 2 + 3\,x$; $36 = 3\,x' + 6\,x$. From the first of these two, we find that $x = 7/3$. Substituting this for x in the second equation, we find that $36 = 3\,x' + 14$, whence $x' = 22/3$.

Problem 6.

a)



b)

We are given a solution f of the differential equation with $f(-1) = 2$. Hence $f'(-1) = -\frac{(-1)(2)^2}{2} = 2$. The equation of the line tangent to the graph of $y = f(x)$ at the point $(-1, 2)$ is therefore $y = 2 + 2(x + 1)$, or $y = 2x + 4$.

c)

We may rewrite the differential equation as $y'(x)/[y(x)]^2 = -x/2$. Thus, $\int_{-1}^{x} y'(t)/[y(t)]^2 \, dt = -\int_{-1}^{x} (t/2)\, dt$, or $[1/y(-1)] - [1/y(x)] = (1/4) - (x^2/4)$. But $y(-1) = 2$, so this equation becomes $(1/y) = (1/2) - [(1 - x^2)/4]$, which simplifies to $y = 4/(1 + x^2)$.