Solutions to the
2006 AP Calculus AB Exam
Free Response Questions

Louis A. Talman
Department of Mathematical & Computer Sciences
Metropolitan State College of Denver

Part A

Problem 1.

■ a)

The curves intersect where $\ln x = x - 2$.

```
a = x /. FindRoot[Log[x] == x - 2, {x, 0.1}][[1]]
```

0.158594339563

```
b = x /. FindRoot[Log[x] == x - 2, {x, 1.6}][[1]]
```

3.14619322062

$$\int_a^b (\text{Log}[x] - (x - 2))\, dx$$

1.94909092741

- **Answer: 1.949**

- **b)**

$$\pi \int_a^b ((\text{Log}[x] - (-3))^2 - ((x - 2) - (-3))^2)\, dx$$

```
34.198613526
```

- **Answer: 34.199**

- **c)**

- **Answer: Either of the integrals $\pi \int_{a-2}^{b-2} [(y+2)^2 - e^{2y}]\, dy$ or $2\pi \int_a^b x\,[\ln x - (x-2)]\, dx$ will do. Evaluation was not required; however**

$$2\pi \int_a^b x\,(\text{Log}[x] - (x - 2))\, dx$$

```
17.099306763
```

and, of course,

$$\pi \int_{a-2}^{b-2} ((y+2)^2 - e^{2y})\, dy$$

```
17.099306763
```

Problem 2:

- **a)**

The rate at which left turns happen is:

$$L[t\_] = 60\sqrt{t}\left(\text{Sin}\left[\frac{t}{3}\right]\right)^2$$

$$60\sqrt{t}\,\text{Sin}\left[\frac{t}{3}\right]^2$$

The number of turns when $0 \le t \le 18$ is therefore

```
NIntegrate[L[t], {t, 0, 18}]
```

```
1657.82373452
```

- **Answer: To the nearest whole number, this is 1658.**

b)

From the graph, we see that $L[t] \geq 150$ on $[a, b]$ where $a \sim 12$ and $b \sim 16$. Solving numerically for a and b, we find

```
a = t /. FindRoot[L[t] == 150, {t, 12}][[1]]
```

```
12.4283095349
```

```
b = t /. FindRoot[L[t] == 150, {t, 16}][[1]]
```

```
16.1216568612
```

$$\frac{1}{b-a}\int_a^b L[t]\,dt$$

```
199.426116196
```

- **Answer: There are 150 or more left turns per hour approximately when $12.428 \leq t \leq 16.122$, where t is measured in hours. The average during this interval is 199.426 left turns per hour.**

c)

During the two-hour interval $13 \leq t \leq 15$,

```
NIntegrate[L[t], {t, 13, 15}]
```

```
431.931400444
```

cars make left turns. 500 oncoming cars pass straight through the intersection in this two-hour period. The product of these two numbers is 215,967.7, and this exceeds the threshhold of 200, 000.

- **Answer: The intersection requires a traffic signal. The reasoning is given in the preceding paragraph.**

Problem 3:

- **a)**

- **Answer:** $g(4) = \int_0^4 f(t)\,dt = 3;$
  $g'(4) = f(4) = 0;$
  $g''(4) = f'(4) = -2.$

- **b)**

- **Answer: By the Fundamental Theorem of Calculus, $g'(x) = f(x)$. Thus, $g'(1) = f(1) = 0$, while $g'(x) = f(x) < 0$ for $0 < x < 1$, and $g'(x) = f(x) > 0$ for $0 < x < 1$. By the First Derivative Test, g has a relative minumum at $x = 1$.**

- **c)**

By periodicity, g(10) - g(5) = g(5) - g(0) = g(5) = 2.Hence, g(10) = g(10) - g(0) = [g(10) - g(5)] + [g(5) - g(0)] = 2 + 2 = 4. Reasoning as in the previous sentence, we find that for any positive integer k, we must have $g(5k) - g(5k-5) = 2$, so that $g(105) = \sum_{k=1}^{21} 2 = 42$. Also, $\int_{105}^{108} f(t)\,dt = \int_0^3 f(t)\,dt = 2$, by inspection of the graph. Hence $g(108) = \int_0^{108} f(t)\,dt = \int_0^{105} f(t)\,dt + \int_{105}^{108} f(t)\,dt = g(105) + \int_{105}^{108} f(t)\,dt = 44$. Also, the first of the following equalities being a consequence of the Fundamental Theorem of Calculus, and the second being a consequence of periodicity, $g'(108) = f(108) = f(3) = 2$. So the equation of the line tangent to $y = g(x)$ at $x = 108$ is $y = 44 + 2(x - 108)$.

- **Answer:** $g(10) = 4$, **and the equation of the tangent line at** $x = 108$ **is** $y = 2x - 172$.

Part B

Problem 4:

- **a)**

- **Answer: Average acceleration is** $\frac{v(80)-v(0)}{80-0} = \frac{49-5}{80} = \frac{11}{20}$ **feet per second per second.**

- **b)**

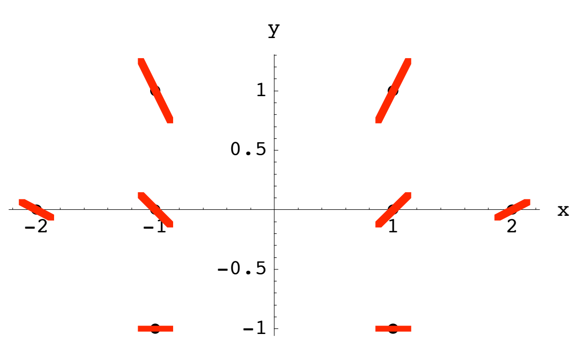
- **Answer:** $\int_{10}^{70} v(t)\,dt$ **measures the distance in feet between the rocket's position at time** $t = 10$ **to its position at time** $t = 70$**. The midpoint Riemann sum with three subdivisions of equal length is** $v(20)\cdot 20 + v(40)\cdot 20 + v(60)\cdot 20 = 440 + 700 + 880 = 2020$ **feet.**

- **c)**

- **Answer: For rocket B, we have, by the Fundamental Theorem of Calculus,** $v(80) = v(0) + \int_0^{80} a(\tau)\,d\tau = 2 + 3\int_0^{80} \frac{d\tau}{\sqrt{\tau+1}} = 50$ **ft/sec. Thus Rocket A is traveling at 49 ft/sec when** $t = 80$**, so rocket B is traveling faster.**

Problem 5:

- **a)**



■ b)

If $y'(x) = [1 + y(x)]/x$, then $y'(x)/[1 + y(x)] = 1/x$. Thus, $\int_{-1}^{x} \{y'(s)/[1 + y(s)]\} \, ds = \int_{-1}^{x} (1/s) \, ds$, as long as $x < 0$ and $y(x) \neq -1$. Consequently, $\ln|1 + y(x)| - \ln|1 + y(-1)| = \ln|x| - \ln|-1|$, or $\ln|1 + y(x)| - \ln 2 = \ln|x|$. Now we have assumed that $x < 0$, so the latter equation is equivalent to $|1 + y(x)| = -2x$. When x is near -1, $y(x)$ must be near 1, so $1 + y(x) > 0$ for such x. Hence, $1 + y(x) = -2x$, or $y(x) = -2x - 1$. We may not have either $x = 0$ or $y(x) = -1$, so the domain of this solution is $-\infty < x < 0$.

■ **Answer:** $\mathbf{y(x) = -2x - 1}$**, when** $\mathbf{-\infty < x < 0}$**.**

Problem 6:

■ a)

If $g(x) = e^{ax} + f(x)$, then $g'(x) = a e^{ax} + f'(x)$, and $g''(x) = a^2 e^{ax} + f''(x)$.

■ **Answer:** $\mathbf{g'(0) = a - 4}$; $\mathbf{g''(0) = a^2 + 3}$.

■ b)

If $h(x) = f(x) \cos kx$, then $h'(x) = f'(x) \cos kx - k f(x) \sin kx$. Thus, $h(0) = 2$, while $h'(0) = f'(0) \cos(k \cdot 0) - k f(0) \sin(k \cdot 0) = -4$.

■ **Answer:** $\mathbf{h'(x) = f'(x) \cos kx - k f(x) \sin kx}$**. The equation of the line tangent to the graph of** $\mathbf{h}$ **at** $\mathbf{x = 0}$ **is** $\mathbf{y = h(0) + h'(0)(x - 0)}$**, or** $\mathbf{y = -4x + 2}$**.**