

Solutions to the 2006 AP Calculus AB Exam (Form B) Free Response Questions

Louis A. Talman
Department of Mathematical & Computer Sciences
Metropolitan State College of Denver

Part A

Problem 1:

■ a)

We must first find the intersection of the curve $y = f(x)$ with the x -axis:

$$\text{In}[1]:= f[x_] = \frac{x^3}{4} - \frac{x^2}{3} - \frac{x}{2} + 3 \cos[x]$$

$$\text{Out}[1]= -\frac{x}{2} - \frac{x^2}{3} + \frac{x^3}{4} + 3 \cos[x]$$

$$\text{In}[2]:= a = x /. \text{FindRoot}[f[x] == 0, \{x, -1.3\}]$$

$$\text{Out}[2]= -1.37312166468$$

The area of the region R is then

$$\text{In}[3]:= \int_a^0 f[x] \, dx$$

$$\text{Out}[3]= 2.9030944245$$

■ **Answer:** To three decimal places, this is **2.903**.

■ **b)**

Using the method of washers, the required volume is

$$\text{In}[4] := \pi \int_a^0 ((f[x] - (-2))^2 - (0 - (-2))^2) dx$$

$$\text{Out}[4] = 59.3614055798$$

■ **Answer:** **59.361**

■ **c)**

We must find the equation of the line tangent to the curve $y = f(x)$ at $(0, 3)$:

$$\text{In}[5] := f'[x]$$

$$\text{Out}[5] = -\frac{1}{2} - \frac{2x}{3} + \frac{3x^2}{4} - 3 \sin[x]$$

$$\text{In}[6] := \% /. x \rightarrow 0$$

$$\text{Out}[6] = -\frac{1}{2}$$

The equation of the tangent line is $y = 3 - x/2$. This line meets the curve when x is slightly bigger than 3:

$$\text{In}[7] := b = x /. \text{FindRoot}[f[x] == 3 - \frac{x}{2}, \{x, 3.0\}]$$

$$\text{Out}[7] = 3.38986766308$$

■ **Answer:** The required integral is $\int_0^b [3 - \frac{x}{2} - f(x)] dx$.

Evaluation is not required. Nevertheless:

$$\text{In}[8] := \int_0^b \left(3 - \frac{x}{2} - f[x]\right) dx$$

$$\text{Out}[8] = 6.98199992437$$

Problem 2:

■ a)

■ **Answer:** The graph of f' is decreasing on the interval $(1.7, 1.9)$, so f is concave downward on that interval.

■ b)

```
In[9]:= c = x /. FindRoot[e-x/4 Sin[x2] == 0, {x, 1.8}]
```

```
Out[9]= 1.77245385091
```

■ **Answer:** With c in place of the number above, f' is positive on the interval $(0, c)$, negative on (c, d) , where d is about 2.5, and positive on $(d, 3]$. By the First Derivative Test, f has a local maximum at $x = c$. The function f has no other local maxima in the interval $(0, 3]$. The area under the first arch of the curve is clearly larger than the area bounded by the second arc of the curve---which, in turn, is clearly larger than the area under the third arch. This means that $f(3) \geq f(c)$ is not possible, nor is $f(0) \geq f(c)$. It follows that $f(x) = \int_0^x f'(t) dt$ has its absolute maximum value for the interval $[0, 3]$ at $x = c \sim 1.772$.

■ c)

The tangent to the graph of f at $x = 2$ has equation $y = y_0 + m(x - 2)$. We must find $m = f'(2)$ and $y_0 = f(2)$.

```
In[10]:= m = e-x/4 Sin[x2] /. x -> 2.0
```

```
Out[10]= -0.459023916751
```

```
In[11]:= y0 = 5 + NIntegrate[e-x/4 Sin[x2], {x, 0, 2}]
```

```
Out[11]= 5.62342672968
```

- **Answer:** Retaining 3 digits' accuracy in the coefficients, the equation of the required tangent line is $y = 5.623 - 0.459(x - 2)$.

Problem 3:

■ a)

- **Answer:** If $y = ax^2$, then $x = 0$ gives both $y = 0$ and $y' = 0$, so that condition (i) is satisfied. However, $y' = 2ax$, and if $x = 4$ this gives $1 = y' = 8a$, by condition (ii), so that $a = 1/8$. But then, using the other part of condition (ii), we find that $1 = y = \frac{1}{8}(4)^2 = 2$, which is not possible. The curve $y = ax^2$ therefore can't satisfy conditions (i) and (ii) with the same choice of a .

■ b)

- **Answer:** Let $g(x) = cx^3 - \frac{x^2}{16}$. Then condition (i) requires that $1 = g(4) = 64c - 1$, so that $c = \frac{1}{32}$. Then $g'(x) = \frac{3}{32}x^2 - \frac{x}{8}$, and $g'(4) = \frac{3}{32} \cdot 16 - \frac{4}{8} = \frac{3}{2} - \frac{1}{2} = 1$. Thus, g satisfies condition (ii) if we take $c = \frac{1}{32}$.

■ c)

- **Answer:** We have $g(x) = \frac{x^3}{32} - \frac{x^2}{16} = \frac{1}{16} \left(\frac{x^3}{2} - x^2 \right)$. Thus, $g'(x) = \frac{1}{16} \left(\frac{3}{2}x^2 - 2x \right) = \frac{x^2}{16} \left(\frac{3}{2}x - 1 \right)$. Consequently, $g'(x) < 0$ for $0 < x < \frac{3}{2}$, and this means that g cannot be increasing over the interval from $x = 0$ to $x = 4$.

■ d)

- Let $h(x) = x^n/k$, where k is a nonzero constant and n is a positive integer. If h meets condition (ii), then $1 = h(4) = 4^n/k$, while $1 = h'(x) = n4^{n-1}/k$ as well. Thus, $k = 4^n$ and $k = n4^{n-1}$. Thus $1 = k/k = 4^n/(n4^{n-1})$, and $n = 4$. But if $n = 4$ and $1 = 4^4/k$, then $k = 4^4 = 256$. Thus, condition (ii) means that $h(x) = x^4/256$. That $h(0) = 0 = h'(0)$ is immediate, and this is condition (i). Also, $h'(x) = x^3/64 > 0$ whenever $0 < x$, and this means h is increasing between $x = 0$ and $x = 4$. So condition (iii) is met.

Problem 4:

■ a)

The point $(22, f(22))$ lies at the midpoint of the segment connecting $(20, 15)$ and $(24, 3)$, which has slope $(15 - 3)/(20 - 24) = -3$. The segment is part of the tangent line to f at $(22, f(22))$.

■ Answer: $f'(22) = -3$ calories per minute per minute.

■ b)

■ Answer: f is increasing only on the intervals $[0, 4]$ and $[12, 16]$. On the latter interval, its rate of increase is $(15 - 9)/(16 - 12) = 3/2$ calories per minute per minute. When $0 \leq x \leq 4$, we have $f'(x) = -\frac{3}{4}x^2 + 3x$; $f''(x) = -\frac{3}{2}x + 3$. Now $f''(x) = 0$ when $x = 2$, $f''(x) > 0$ when $0 < x < 2$, and $f''(x) < 0$ when $2 < x < 4$. Thus f' is an increasing function on the interval $[0, 2]$, and is a decreasing function on the interval $[2, 4]$. It follows that f' has a maximum for the interval $[0, 4]$ when $x = 2$, and this maximum value is $f'(2) = 3$. This is larger than $f'(x)$ when $12 < x < 16$, so the maximal value of $f'(x)$, i.e., the maximal rate of increase for the rate at which calories are burned is 3 calories per minute per minute at time $x = 2$.

■ c)

The total number of calories burned over the time interval $6 \leq t \leq 18$ is $\int_6^{18} f(t) dt$. Computing the areas of the relevant rectangles and the relevant trapezoid, we find that this is $6 \cdot 9 + 4 \cdot (15 + 9)/2 + 2 \cdot 15 = 132$ calories.

■ Answer: 132 calories.

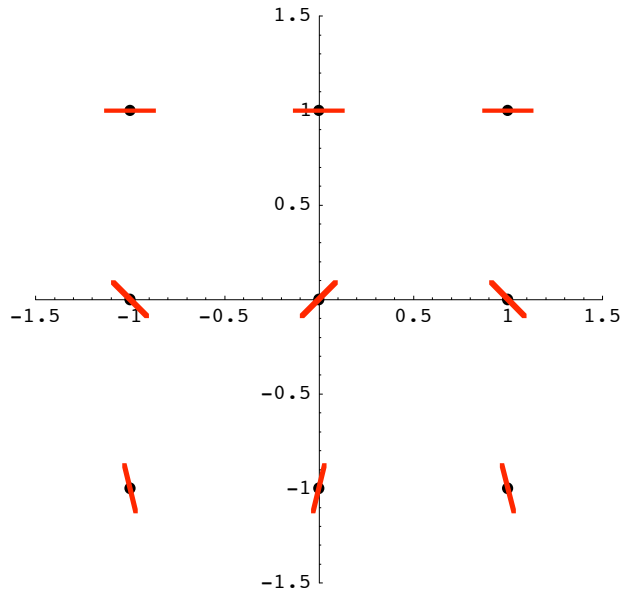
■ d)

We must have $\frac{1}{18-6} \int_6^{18} [f(t) + c] dt = 15$. But $\frac{1}{12} \int_6^{18} [f(t) + c] dt = \frac{1}{12} \int_6^{18} f(t) dt + \frac{1}{12} \int_6^{18} c dt = \frac{132}{12} + c = 11 + c$. Thus, $11 + c = 15$, and $c = 4$.

■ Answer: $c = 4$.

Problem 5:

■ a)



- Graphics -

■ b)

If $y = c$ is a solution, then y' is identically zero. Because $y' = (y - 1)^2 \cos(\pi x)$, this means that $y = 1$ is the required constant solution.

■ Answer: $y \equiv 1$.

■ c)

If $y'(x) = [y(x) - 1]^2 \cos(\pi x)$, then $y'(x)/[y(x) - 1] = \cos(\pi x)$. Thus, $\int_0^x y'(s)/[y(s) - 1]^2 ds = \int_0^x \cos(\pi s) ds$, or $\frac{1}{1-y(x)} - \frac{1}{1-y(1)} = \frac{1}{\pi} \sin(\pi x) - \frac{1}{\pi} \sin(\pi \cdot 0)$. But $y(1) = 0$ is given, so $\frac{1}{1-y(x)} = \frac{1}{\pi} \sin(\pi x) + 1$, whence $y(x) = \sin(\pi x)/[\pi + \sin(\pi x)]$.

- Answer: $f(x) = \sin(\pi x) / [\pi + \sin(\pi x)]$.

Problem 6:

- a)

- Answer: The integral $\int_{30}^{60} |v(t)| dt$ gives the total distance that the car travels during the interval from $t = 30$ to $t = 60$. By the Trapezoid Rule, we have

$$\int_{30}^{60} |v(t)| dt \sim \frac{(14+10)}{2} \cdot 5 + \frac{(10+0)}{2} \cdot 15 + \frac{(0+10)}{2} \cdot 10 = 185 \text{ ft/sec.}$$

- b)

- Answer: The integral $\int_0^{30} a(t) dt$ gives the change in the car's velocity from $t = 0$ to $t = 30$. This is $v(30) - v(0) = (-14) - (-20) = 6 \text{ ft/sec.}$

- c)

- Answer: $v(0) = -20$, while $v(60) = 10$, so $v(0) < -5 < v(60)$. We are given that v is continuous on $[0, 60]$, so by the Intermediate Value Theorem there must be a time $t_0 \in (0, 60)$ such that $v(t_0) = -5$.

- d)

- Answer: $v(0) = -20$ and $v(25) = -20$. v and $a = v'$ are given continuous throughout $[0, 60]$, which contains $[0, 25]$. Thus, by the Mean Value Theorem, there is $t_1 \in (0, 25)$ such that $a[t_1] = v'[t_1] = [v(25) - v(0)] / (25 - 0) = 0$. Because t_1 lies between 0 and 25, it lies, *a fortiori*, between 0 and 60.