Solutions to the
2007 AP Calculus AB Exam
Free Response Questions

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Part A

Problem 1.

a)

The limits of the integral that gives the area are the solutions of the equation $2 = \frac{20}{1+x^2}$:

In[5]:= $\text{Solve}\left[2 == \frac{20}{1+x^2}, x\right]$

Out[5]= $\{\{x \to -3\}, \{x \to 3\}\}$

The required area is therefore

In[6]:= $\int_{-3}^{3} \left(\frac{20}{1+x^2} - 2\right) dx$

Out[6]= $-12 + 40\,\text{ArcTan}[3]$

The required area is $40 \arctan 3 - 12$.

b)

Using the method of washers, we find that the required volume is

In[7]:= $\pi \int_{-3}^{3} \left(\left(\frac{20}{1+x^2}\right)^2 - 2^2\right) dx$

Out[7]= $\pi\,(96 + 400\,\text{ArcTan}[3])$

The required volume is $\pi\,(96 + 400 \arctan 3)$.

c)

The diameter of the semicircle at $x = t$ is $\frac{20}{1+t^2} - 2$, so the radius is $\frac{10}{1+t^2} - 1$. Hence the area $A(t)$ of the cross section at $x = t$ is $A(t) = \frac{\pi}{2}\left(\frac{10}{1+t^2} - 1\right)^2$. The required volume is therefore

In[8]:= $\frac{\pi}{2} \int_{-3}^{3} \left(\frac{10}{1 + t^2} - 1\right)^2 dt$

Out[8]= $\frac{1}{2} \pi (36 + 60 \text{ArcTan}[3])$

The required volume is $\frac{\pi}{2}(36 + 60 \arctan 3)$.

Problem 2

a)

The amount of water that enters the tank during the time interval $0 \le t \le 7$ is $\int_0^7 f(t)\, dt = \int_0^7 \left[100\, t^2 \sin \sqrt{t}\,\right] dt$:

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In[9]:= NIntegrate[100 t^2 Sin[Sqrt[t]], {t, 0, 7}]

Out[9]= 8263.80654266
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That's about 8263.806 gallons, or, to the nearest gallon, 8264 gallons.

b)

From the graph and what we are given about the intersection points of the curves, we see that the rate at which water leaves the tank exceeds that at which it leaves the tank on the intervals $[0,\ 1.617)$ and $(3,\ 5.076)$. It follows that the amount of water in the tank is decreasing on each of the intervals $[0,\ 1.617]$ and $[3,\ 5.076]$.

c)

The rate at which the amount of water in the tank increases is, as we have seen in part b), negative on the interval $(3,\ 5.076)$. It is positive on the interval $(1.617,\ 3)$. By the First Derivative Test, the amount of water in the tank has a local maximum at $t = 3$. We must also consider the amount of water in the tank when $t = 0$ and the amount when $t = 7$. Over the interval $[0,\ 7]$, $3 \times 250 + 4 \times 2000 = 8750$ gallons of water have left the tank, while, according to part a), 8263.806 gallons have entered. When $t = 7$, the amount of water in the tank is therefore $5000 + 8263.806 - 8750 = 4513.806$ gallons. The amount of water in the tank at $t = 3$ is $5000 + \int_0^{3.000} \left[100\, t^2 \sin \sqrt{t} - 250\right] dt$:

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In[10]:= 5000 + NIntegrate[100 t^2 Sin[Sqrt[t]] - 250, {t, 0, 3}]

Out[10]= 5126.59080072
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The maximum occurs when $t = 3$ and, to the nearest gallon, is 5127 gallons.

Problem 3

a)

The functions f and g are differentiable for all real numbers, and therefore are continuous everywhere. As a composition of continuous differentiable functions, h must be everywhere continuous and everywhere differentiable. If $h(x) = f[g(x)] - 6$, then $h(1) = f[g(1)] - 6 = f(2) - 6 = 9 - 6 = 3$, while $h(3) = f[g(3)] - 6 = f(4) - 6 = -1 - 6 = -7$. But -5 lies between $h(1) = 3$ and $h(3) = -7$. By the Intermediate Value Property of continuous functions, there must be $r \in (1, 3)$ where $h(r) = -5$.

b)

As noted above, the function h is everywhere differentiable. By the Mean Value Theorem, there must be a $c \in (1, 3)$ where $h'(c)(3 - 1) = h(3) - h(1) = -7 - 3 = -10$, or $h'(c) = -5$.

c)

If w is the function given by $w(x) = \int_1^{g(x)} f(t)\,dt$, then by the Fundamental Theorem of Calculus and the Chain Rule, $w'(x) = f[g(x)]\,g'(x)$. Thus, $w'(3) = f[g(3)]\,g'(3) = f(4)\,g'(3) = (-1)\cdot 2 = -2$.

d)

Putting $F(x) = g^{-1}(x)$, we have $F'(x) = 1\big/g'\left[g^{-1}(x)\right]$. Thus, $F'(2) = 1\big/g'\left[g^{-1}(2)\right] = 1/g'(1) = 1/5$. The equation of the tangent line is $y = 1 + \frac{1}{5}(x - 2)$.

Problem 4

a)

The particle is farthest to the left when the function $x(t) = e^{-t}\sin t$ assumes its absolute minimum value on the interval $[0, 2\pi]$. Because $x(t) \geq 0$ when $0 \leq t \leq \pi$ and when $t = 2\pi$, this minimum must lie somewhere in $(\pi, 2\pi)$, and must therefore be at a critical point of x. We have $x'(t) = e^{-t}(\cos t - \sin t)$, and e^{-t} never vanishes. Hence the critical point of $x'(t)$ that we seek must lie at $t = 5\pi/4$, which is the only zero of $\cos t - \sin t$ in the interval $(\pi, 2\pi)$. We conclude that the particle is farthest left when $t = 5\pi/4$.

b)

Because $x'(t) = e^{-t}(\cos t - \sin t)$ and $x''(t) = -2e^{-t}\cos t$, the equation $A\,x''(t) + x'(t) + x(t) = 0$ reduces to $(2A - 1)\cos t = 0$. This latter equation is true for $0 < t < 2\pi$ only if $2A - 1 = 0$, or if $A = 1/2$.

Problem 5

■ a)

The linearization of r at $t = 5$ is the linear function $L(t) = r(5) + r'(5)(t - 5) = 30 + 2(t - 5)$. An approximate value for $r(5.4)$ is $L(5.4) = 30 + 2(5.4 - 5) =$ 30.8 feet. The curve $r = r(t)$ is given to be concave downward, so the tangent line at each point lies (locally) above the curve. Thus, our estimate of 30.8 ft is an overestimate.

■ b)

Because $V(t) = \frac{4}{3}\pi[r(t)]^3$, we have $V'(t) = 4\pi[r(t)]^2 r'(t)$. Thus, $V'(5) = 4\pi\, 30^2 \cdot 2.0 \sim 22,619.467\ \text{ft}^3/\text{min}$.

■ c)

The right Riemann sum corresponding to the data given is $4.0(2 - 0) + 2.0(5 - 2) + 1.2(7 - 5) + 0.6(11 - 7) + 0.5(12 - 11) = 19.3$ feet. By the Fundamental Theorem of Calculus, $\int_0^{12} r'(t)\,dt = r(12) - r(0)$ is the change, in feet, in the radius from its value when $t = 0$ to its value when $t = 12$.

■ d)

The function r is given concave down, so r' is a decreasing function. Consequently, $r'(t) \geq r'(b)$ when t lies anywhere in an interval $[a, b]$. It follows that each term of the right Riemann sum is less than the area under the curve in the corresponding interval. The right Riemann sum therefore underestimates the integral.

Problem 6

■ a)

If $f(x) = k\sqrt{x} - \ln x$, then $f'(x) = \frac{k}{2\sqrt{x}} - \frac{1}{x}$, and $f''(x) = -\frac{k}{4x\sqrt{x}} + \frac{1}{x^2}$.

■ b)

We must have $0 = f'(1) = \frac{k}{2} - 1$, or $k = 2$ if $x = 1$ is to be a critical point. Then $f''(1) = \frac{-1}{2} + 1 = \frac{1}{2} > 0$, so by the Second Derivative Test the critical point gives a relative minimum.

■ c)

An inflection point lies where $f''(x)$ changes sign. Because f'' is continuous on the interval in question, we must have $f''(x) = 0$ at such a point. Thus, $-\frac{k}{4x\sqrt{x}} + \frac{1}{x^2} = 0$, or $x = \frac{16}{k^2}$. But we want our inflection point to lie on the x-axis, and this means we must have $f\left(\frac{16}{k^2}\right) = 0$, or $4 - \ln\frac{16}{k^2} = 0$. This is equivalent to $\ln\frac{4}{k} = 2$, whence $k = 4/e^2$.