Solutions to the 2007 AP Calculus AB (Form B) Exam Free Response Questions

Louis A. Talman
Department of Mathematical & Computer Sciences
Metropolitan State College of Denver

Part A

Problem 1

■ a)

The curve $y = e^{2x-x^2}$ intersects the line $y = 2$ where $2x - x^2 = \ln 2$:

```
In[1]:= Solve[2 x - x^2 == Log[2], x]

Out[1]= {{x → 1 - Sqrt[1 - Log[2]]}, {x → 1 + Sqrt[1 - Log[2]]}}
```

Thus, the area of the region R is $\int_a^b \left(e^{2x-x^2} - 2\right) dx$, where $a = 1 - \sqrt{1 - \ln 2}$ and $b = 1 + \sqrt{1 - \ln 2}$. The integral must be done numerically:

```
In[2]:= NIntegrate[e^(2 x - x^2) - 2, {x, 1 - Sqrt[1 - Log[2]], 1 + Sqrt[1 - Log[2]]}]

Out[2]= 0.51414278561
```

The area of region R is, to three digits to the right of the decimal, 0.514.

■ b)

The curve $y = e^{2x-x^2}$ intersects the line $y = 1$ where $2x - x^2 = \ln 1 = 0$, or at $x = 0$ and at $x = 2$. Thus, the sum of the areas of the regions R and S is the integral $\int_0^2 \left(e^{2x-x^2} - 1\right) dx$. From this we subtract the integral of Part a) to obtain the area of the region S. Once again, we must integrate numerically:

```
In[3]:= NIntegrate[e^(2 x - x^2) - 1, {x, 0, 2}] -
         NIntegrate[e^(2 x - x^2) - 2, {x, 1 - √(1 - Log[2]), 1 + √(1 - Log[2])}]
Out[3]= 1.54601415295
```

The area of region S is, to three digits to the right of the decimal, 1.546.

■ c)

Using the method of washers, the volume of the solid generated when the region R is rotated about the line $y = 1$ is $\pi \int_a^b \left[\left(e^{2x - x^2} - 1\right)^2 - 1\right] dx$, where $a = 1 - \sqrt{1 - \ln 2}$, and $b = 1 + \sqrt{1 - \ln 2}$.

Evaluation of this integral is not required, but let's do it numerically anyway:

```
In[4]:= π NIntegrate[(e^(2 x - x^2) - 1)^2 - 1, {x, 1 - √(1 - Log[2]), 1 + √(1 - Log[2])}]
Out[4]= 4.1466068494
```

Problem 2

■ a)

Acceleration is the derivative, taken with respect to time, of velocity.

```
In[5]:= v[t_] = Sin[t^2]
Out[5]= Sin[t^2]
In[6]:= v'[3]
Out[6]= 6 Cos[9]
```

Acceleration at time $t = 3$ is $6 \cos 9$. No units are given in the problem, so we give none. However, acceleration must be in units of length / time^2.

■ b)

Total distance traveled is the integral of the magnitude of velocity over the time interval in question. We integrate numerically:

```
In[7]:= NIntegrate[Sin[t^2], {t, 0, √π}] +
         NIntegrate[-Sin[t^2], {t, √π, √(2 π)}] + NIntegrate[Sin[t^2], {t, √(2 π), 3}]
Out[7]= 1.70241001652
```

The required distance is, to three digits to the right of the decimal, 1.702.

c)

Change in position is the integral of velocity over the time interval in question. We adjust by adding the initial position in order to obtain final position. Once again, integrating numerically:

```
In[8]:= 5 + NIntegrate[Sin[t^2], {t, 0, 3}]
Out[8]= 5.77356252689
```

When $t = 3$, the particle is at $x = 5.774$, correct to three digits to the right of the decimal.

d)

At the instant when the particle is farthest to the right, it must be at an end-point or we must have $v(t) = 0$, for otherwise the particle is moving—either rightward or leftward. In fact, velocity must change from positive to negative at such an instant, for the particle must have just stopped rightward motion and be about to begin leftward motion. There are three points to consider in the given interval: $t = \sqrt{\pi}$, $t = \sqrt{3\pi}$, and $t = \sqrt{5\pi}$. (We reject $t = 0$ on the ground that the particle moves rightward in the interval immediately to the right of $t = 0$.) We could argue on the basis of areas in the diagram, but it's easier just to do the numerical integrations and compare the results:

```
In[9]:= Table[5 + NIntegrate[Sin[t^2], {t, 0, Sqrt[k π]}], {k, 1, 5, 2}]
Out[9]= {5.89483146948, 5.78825896557, 5.75244266745}
```

The particle is farthest right when $t = \sqrt{\pi}$.

Problem 3

a)

We are given $W(v) = 55.6 - 22.1\, v^{0.16}$.

```
In[10]:= W[v_] = 55.6 - 22.1 v^0.16
Out[10]= 55.6 - 22.1 v^0.16
In[11]:= W'[20]
Out[11]= -0.285526919601
```

$W'(20) = -0.285527\,\text{deg}/\text{mi}/\text{hr}$, so that for small changes in wind velocity, the approximate change in wind chill is -0.286 degrees Fahrenheit for each mile per hour change in wind velocity.

b)

Averate rate of change of W over the interval from $v = 5$ to $v = 60$ is $[W(60) - W(5)]/[60 - 5]$:

```
In[12]:= (W[60] - W[5])/(60 - 5)
Out[12]= -0.253796734102
```

The required average rate of change is, correct to three digits to the right of the decimal, -0.254. We solve the equation $W'(v) = [W(60) - W(5)]/[60 - 5]$ numerically to find the value of v at which instantaneous rate of change is the same as the average rate of change:

In[13]:= `NSolve[W'[v] == (W[60] - W[5])/(60 - 5), v]`

Out[13]= `{{v → 23.0110260815}}`

The wind velocity at which the instantaneous rate of change is equal to the average rate of change is $v = 23.011$. Note: Although we have given three digits to the right of the decimal, the precision of the numbers given in the statement of the problem makes this level of precision to be of doubtful merit in the answer.

■ c)

We put $v[t] = 20 + 5t$. Then

In[14]:= `v[t_] = 20 + 5 t`

Out[14]= `20 + 5 t`

In[15]:= `D[W[v[t]], t]`

Out[15]= $-\frac{17.68}{(20+5\,t)^{0.84}}$

Evaluating this expression when $t = 3$ gives

In[16]:= `% /. t → 3`

Out[16]= `-0.892205925875`

When $t = 3$, the rate of change of wind chlll with respect to time is -0.892 deg F/hr.

Part B

Problem 4

■ a)

Local maxima are to be found at points where the derivative undergoes a sign change from positive to negative. For this function, that happens when $x = -3$ and when $x = 4$.

■ b)

Any point where the derivative changes from increasing to decreasing, or vice versa, is an inflection point. For this function, we find such points at $x = -4$, $x = -1$, and $x = 2$.

c)

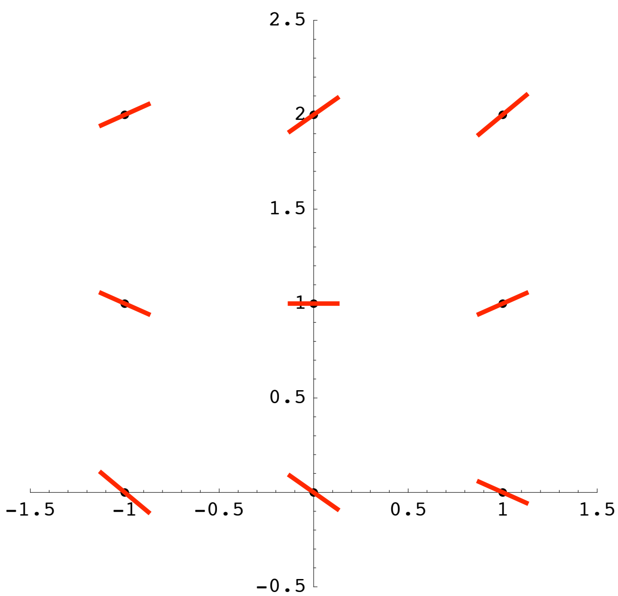
If the derivative is both positive and increasing, then the function is concave upward and has positive slope. This function displays such behavior on the intervals $(-5, -4)$ and $(1, 2)$. (Whether or not to include some of the end-points depends on how we define "concave upward". This varies with textbook.)

d)

The absolute minimum value of $f(x)$ over the interval $-5 \leq x \leq 5$ is $f(1) = 3$. We know that $x = 1$ gives a local minimum because the derivative changes sign from negative to positive at $x = 1$. The only other possibilities for minima are at the end-points of the interval, because we have already [Part b)] identified the other critical points in the interval as the locations of relative maxima. We have $f(-5) = 3 + \int_1^{-5} f'(x)\,dx = 3 - \pi/2 + 2\pi = 3 + 3\pi/2 > 3$. (The integral is the sum of the signed areas of two semicircles.) On the other hand, $f(5) = 3 + \int_1^5 f'(x)\,dx = 3 + \frac{1}{2}3 \cdot 2 - \frac{1}{2}1 \cdot 1 = 11/2 > 3$. (This integral is the sum of the signed areas of two triangles.) It follows that the absolute minimum value of x over the interval $-5 \leq x \leq 5$ is $f(1) = 3$.

Problem 5

a)



b)

Because $y' = \frac{1}{2}x + y - 1$, we have $y'' = \frac{1}{2} + y' = \frac{1}{2}x + y - \frac{1}{2}$. Solutions to the original differential equation must be concave upward in the region where $y'' > 0$, or where $\frac{1}{2}x + y - \frac{1}{2} > 0$. This is the region lying above the line $y = -\frac{1}{2}x + \frac{1}{2}$, which is the line that passes through the point with coordinates $(0, 1/2)$ and the point with coordinates $(1, 0)$,

c)

If $y = f(x)$ is the solution of this differential equation for which $f(0) = 1$, then $f'(0) = \frac{1}{2}\cdot 0 + 1 - 1 = 0$, so f has a critical point at $x = 0$. From Part b), we have $f''(0) = \frac{1}{2}\cdot 0 + 1 - \frac{1}{2} = \frac{1}{2} > 0$. It then follows from the Second Derivative Test that f has a relative minimum at $x = 0$.

■ d)

If $y = m x + b$ is to be a solution, then $y' = m$, and it follows from substituting m for y' and $m x + b$ for y in the differential equation that we must have $m = \frac{1}{2} x + y - 1 = \frac{1}{2} x + m x + b - 1$ for all x. This is possible only if $m + \frac{1}{2} = 0$ and $b - 1 = m$. Hence $m = -1/2$ and $b = 1/2$.

Problem 6

■ a)

The function f is twice differentiable throughout the interval in question, and so is continuous there. By the Mean Value Theorem, there is a number c, $2 < c < 5$, such that $f'(c) = [f(5) - f(2)]/(5 - 2) = [2 - 5]/[5 - 2] = -1$.

■ b)

We have $g'(x) = f'[f[x]]\,f'(x)$. Thus, $g'(2) = f'[f(2)]\,f'[2] = f'[5]\,f'[2]$, while $g'(5) = f'[f(5)]\,f'(5) = f'[2]\,f'(5)$. Thus $g'(2) = g'(5)$, and it now follows from Rolle's Theorem that there is a number k in the interval $(2, 5)$ where $g'(k) = 0$.

■ c)

The second derivative of g is given by $g''(x) = f''[f(x)]\cdot[f'(x)]^2 + f'[f(x)]\cdot f''[x]$. It therefore follows from the hypothesis that f'' vanishes identically that g'' also vanishes identically. Because inflection points are to be found where the second derivative changes sign, g can have no such points.

■ d)

If $h(x) = f(x) - x$, then $h(2) = f(2) - 2 = 5 - 2 = 3 > 0$, while $h(5) = f(5) - 5 = 2 - 5 = -3 < 0$. But f is given twice differentiable, so f must be continuous. Consequently, h, which is the difference of two continuous functions, is also continuous. By the Intermediate Value Property of continuous functions, there must be a point r in $(2, 5)$ where $h(r) = 0$, because $h(2) > 0$ while $h(5) < 0$.