

Solutions to the 2008 AP Calculus AB Exam Free Response Questions

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Part A

Problem 1.

■ a)

The area of the given region R is $\int_0^2 [\sin(\pi x) - (x^3 - 4x)] dx$.

$$\int_0^2 (\sin[\pi x] - (x^3 - 4x)) dx$$

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■ b)

We must first solve the equation $x^3 - 4x = -2$ to find the limits of integration:

Solve $[x^3 - 4x = -2, x]$

$$\left\{ \left\{ x \rightarrow \frac{(-9 + i\sqrt{111})^{1/3}}{3^{2/3}} + \frac{4}{(3(-9 + i\sqrt{111}))^{1/3}} \right\}, \right.$$

$$\left\{ x \rightarrow -\frac{(1 + i\sqrt{3})(-9 + i\sqrt{111})^{1/3}}{2 \cdot 3^{2/3}} - \frac{2(1 - i\sqrt{3})}{(3(-9 + i\sqrt{111}))^{1/3}} \right\},$$

$$\left\{ x \rightarrow -\frac{(1 - i\sqrt{3})(-9 + i\sqrt{111})^{1/3}}{2 \cdot 3^{2/3}} - \frac{2(1 + i\sqrt{3})}{(3(-9 + i\sqrt{111}))^{1/3}} \right\}$$

N[%]

$$\left\{ \left\{ x \rightarrow 1.6751309 - 1.110223 \times 10^{-16} i \right\}, \left\{ x \rightarrow 0.53918887 + 0. i \right\}, \left\{ x \rightarrow -2.2143197 - 5.5511151 \times 10^{-17} i \right\} \right\}$$

Although the solutions appear to be complex, the imaginary parts are very tiny and result from numerical errors. We need the

roots that lie in the interval $[0, 2]$, so we will ignore the third solution.

The area of that part of the region R which lies below the horizontal line $y = -2$ is given by $\int_{0.539}^{1.675} [-2 - (x^3 - 4x)] dx$.

■ c)

The area of a cross-section of the solid perpendicular to the x -axis at $x = t$ is $[\sin(\pi t) - (t^3 - 4t)]^2$. Thus, the volume of the solid is $\int_0^2 [\sin(\pi t) - (t^3 - 4t)]^2 dt$:

$$\int_0^2 (\sin[\pi t] - (t^3 - 4t))^2 dt$$

$$\frac{1129}{105} - \frac{24}{\pi^3}$$

Numerically, this is

$$\mathbf{N[\%]}$$

$$9.9783441$$

■ d)

Under the conditions given, the pool is a region in 3-space whose base is R and whose cross-section perpendicular to the x -axis at $x = t$ has area $[\sin(\pi t) - (t^3 - 4t)](3 - t)$. The volume is thus $\int_0^2 [\sin(\pi t) - (t^3 - 4t)](3 - t) dt$:

$$\int_0^2 (\sin[\pi t] - (t^3 - 4t))(3 - t) dt$$

$$\frac{116}{15} + \frac{2}{\pi}$$

Numerically,

$$\mathbf{N[\%]}$$

$$8.3699531$$

Problem 2.

■ a)

At 5:30pm, the rate at which the number of people standing in line was changing was approximately $[L(7) - L(4)]/(7 - 4) = (150 - 126)/(7 - 4) = 8$ people per hour.

■ b)

The average number of people standing in line during the first four hours that tickets were on sale was $\frac{1}{4-0} \int_0^4 L(t) dt$. From the data in the table and the trapezoid rule, that is approximately

$$\frac{1}{4-0} \left(\frac{120+156}{2} (1-0) + \frac{156+176}{2} (3-1) + \frac{176+126}{2} (4-3) \right)$$

$$\frac{621}{4}$$

■ c)

By the Mean Value Theorem, there must be a point $\xi_1 \in (1, 3)$ such that $L'(\xi_1) = \frac{L(3)-L(1)}{3-1} = \frac{176-156}{2} > 0$ and there must be a point $\xi_2 \in (3, 4)$ such that $L'(\xi_2) = \frac{L(4)-L(3)}{4-3} = \frac{126-176}{1} < 0$. Now $L''(t)$ is defined for all $t \in [0, 9]$, and this means that L' must be a continuous function on $[0, 9]$. By the Intermediate Value Theorem for Continuous Functions, there must be a point $\eta_1 \in (\xi_1, \xi_2)$ where $L'(\eta_1) = 0$. By similar reasoning, there must be $\xi_3 \in (4, 7)$ where $L'(\xi_3) > 0$, and so $\eta_2 \in (\xi_2, \xi_3)$ where $L'(\eta_2) = 0$. Further, there must be $\xi_4 \in (7, 8)$ for which $L'(\xi_4) < 0$, and therefore $\eta_3 \in (\xi_3, \xi_4)$ where $L'(\eta_3) = 0$.

We conclude that $L'(t)$ takes on the value 0 at least 3 times in the interval $(0, 9)$.

Note: It actually suffices to know that L is differentiable throughout $(0, 9)$ to make this argument: Derivatives have the Intermediate Value Property even when they are not continuous.

■ d)

If $T(t)$ denotes the number of tickets sold at time t , we are given $T(0) = 0$ and $T'(t) = 550 t e^{-t/2}$. Consequently,

$$T[t_0] = \int_0^{t_0} 550 t e^{-t/2} dt$$

$$550 (4 - 2 e^{-t/2} (2 + t))$$

Thus,

$$T[3.0]$$

$$972.78412$$

To the nearest whole number, 973 tickets have been sold by 3:00pm.

Problem 3.

■ a)

We write $V(t) = \pi [r(t)]^2 h(t)$, and find that $V'(t) = 2\pi r(t) h(t) r'(t) + \pi [r(t)]^2 h'(t)$. We are given that $V'(t) = 2000$ cc/min for all t , and that when $t = t_0$ we have $r(t_0) = 100$ cm, $h(t_0) = 0.5$ cm, and $r'(t_0) = 2.5$ cm/min. Thus

$2000 = 2\pi \cdot 100 \cdot 0.5 \cdot 2.5 + \pi 100^2 \cdot h'(t_0)$, so that

$$\text{Solve}[2000 == 2\pi 100 \times 0.5 \times 2.5 + \pi 100^2 h'[t_0], h'[t_0]]$$

$$\{\{h'[t_0] \rightarrow 0.038661977\}\}$$

At the given instant, $h'(t_0) = 0.039$ cm/min.

■ **b)**

Taking $t = 0$ to be the moment when the recovery device goes into action, we have $V'(t) = 2000 - 400\sqrt{t}$. Thus, $V(t)$ has a critical point at $t = 25$, when $V'(t) = 0$. Because $V'(t) > 0$ for $t < 25$, while $V'(t) < 0$ when $t > 25$, it follows from the First Derivative Test that $V(t)$ is maximal when $t = 25$.

■ **c)**

If there were 60,000 cc of oil in the slick at the moment $t = 0$, when the recovery device began to operate, then by the Fundamental Theorem of Calculus we must have $V(t) - 60\,000 = V(t) - V(0) = \int_0^t V'(\tau) d\tau$. From what we saw in part b) above, we must therefore have $V(t) = 60\,000 + \int_0^t (2000 - 400\sqrt{\tau}) d\tau$.

Part B

Problem 4.

■ **a)**

By the Fundamental Theorem of Calculus and what we are given, we must have $x(t) = -2 + \int_0^t v(\tau) d\tau$. This means that $v(3) = -10$, $v(5) = -7$, and $v(6) = -9$. From the figure and the information given, we have $x'(t) = v(t) < 0$ for $0 < t < 3$ and for $5 < t < 6$, while $x'(t) > 0$ for $3 < t < 5$. Thus x is decreasing when $0 \leq t \leq 3$ and when $5 \leq t \leq 6$, while x is increasing when $3 \leq t \leq 5$. Thus, the particle is furthest to the left when $t = 3$ and its position at that instant is $x = -10$.

■ **b)**

Because $x(0) = -2$ and $x(3) = -10$, (see part a)), the particle moves through $x = -8$ once (leftward bound) when $0 < t < 3$. Because $x(3) = -10$, and $x(5) = -7$ (see part a)), it moves through $x = -8$ again (rightward bound) at some time in the interval $(3, 5)$. Because $x(5) = -7$ and $x(6) = -9$ (see part a) again) it moves through $x = -8$ still again (now leftward bound) at some time in the interval $(5, 6)$. The existence of these times is guaranteed, in each case, because the differentiable function x must be continuous on $[0, 6]$, and continuous functions have the intermediate value property. That these three instances are the only instances is guaranteed by the fact that x must be monotonic on each of the intervals $[0, 3]$, $[3, 5]$, and $[5, 6]$. We conclude that the particle passes through $x = -8$ just three times.

■ **c)**

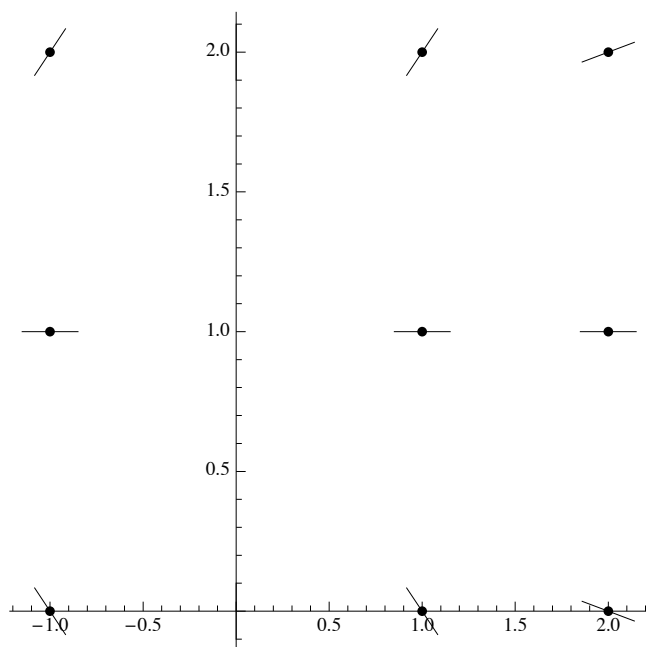
Speed is the magnitude of velocity. On the interval $(2, 3)$, velocity is—from the graph—increasing, but negative. On this interval, we obtain the magnitude of velocity by reflecting the relevant portion of the curve about the t -axis. So speed is decreasing on $(2, 3)$.

■ d)

Acceleration is $v'(t)$. Thus, acceleration is negative on intervals where $v(t)$ is decreasing. From the graph and what we have been given about it, **acceleration is negative on $[0, 1)$ and on $(4, 6]$.**

Problem 5.

■ a)



■ b)

We have $y'(x)/[y(x) - 1] = 1/x^2$, with $y(2) = 0$. Hence $\int_2^x [y'(\tau)/[y(\tau) - 1]] d\tau = \int_2^x \tau^{-2} d\tau$. Integrating, we find that $\ln[1 - y(x)] - \ln[1 - y(2)] = \frac{1}{2} - \frac{1}{x}$. But $y(2) = 0$ is given, so $\ln[1 - y(x)] = \frac{1}{2} - \frac{1}{x}$. Thus, $1 - y(x) = \exp\left(\frac{1}{2} - \frac{1}{x}\right)$, or **$y = 1 - \sqrt{e} / e^{1/x}$.**

■ c)

$$\lim_{x \rightarrow \infty} y(x) = \lim_{x \rightarrow \infty} \left(1 - \sqrt{e} / e^{1/x}\right) = 1 - \sqrt{e} / \lim_{x \rightarrow \infty} e^{1/x} = 1 - e^{1/2}.$$

Problem 6.

■ a)

$f'(e^2) = (1 - \ln e^2)/(e^2)^2 = (1 - 2)/e^4 = -e^{-4}$, so an equation of the tangent line at the point which corresponds to $x = e^2$ is $y = 2e^{-2} - e^{-4}(x - e^2)$.

■ b)

$f'(x) = 0$ when $(1 - \ln x)/x^2 = 0$, or when $x = e$, so the x -coordinate of the critical point of f is $x = e$. We note that $f'(x) > 0$ when $x < e$, but that $f'(x) < 0$ when $x > e$, and it follows that f has a local maximum at $x = e$.

■ c)

From $f'(x) = (1 - \ln x)/x^2$, we have $f''(x) = (2 \ln x - 3)/x^3$. We note that $f''(x) < 0$ when $x < e^{3/2}$ but that $f''(x) > 0$ when $x > e^{3/2}$. The function f therefore has an inflection point at $x = e^{3/2}$.

■ d)

$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{\ln x}{x} = -\infty$.