Solutions to the 2008 AP Calculus AB Exam Free Response Questions (Form B)

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Part A

Problem 1.

■ a)

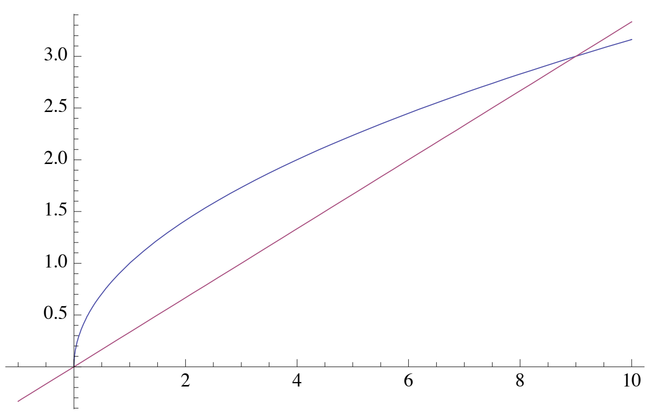
We must find the intersection points:

```
Solve[√x == x/3, x]
```

```
{{x → 0}, {x → 9}}
```

And here is the graph:

```
Plot[{√x, x/3}, {x, -1, 10}]
```



$$\int_0^9 \left(\sqrt{x} - \frac{x}{3}\right) dx$$

$$\frac{9}{2}$$

The area of the region is $\frac{9}{2}$.

b

By the method of shells:

$$2\pi \int_0^9 \left(\sqrt{x} - \frac{x}{3}\right)(x+1)\, dx$$

$$\frac{207\pi}{5}$$

or, by the method of washers:

$$\pi \int_0^3 \left((3y+1)^2 - \left(y^2+1\right)^2\right) dy$$

$$\frac{207\pi}{5}$$

The volume of the solid of revolution is $207\pi/5$.

c

Rewriting the equations of the curve in terms of y, we find that $y = \sqrt{x}$ becomes $x = y^2$, while $y = x/3$ becomes $x = 3y$. The area of a cross-section perpendicular to the y-axis is therefore $\left(3y - y^2\right)^2$.

$$\int_0^3 \left(3y - y^2\right)^2 dy$$

$$\frac{81}{10}$$

The volume of the solid is $81/10$.

Problem 2

a)

We integrate velocity to obtain distance traveled. (The problem gives speed rather than velocity, but the given speed is never zero, and this guarantees that travel is unidirectional. We take the positive direction to be the direction of travel.) The integration $120\int_0^2\left(1 - e^{-10t^2}\right)dt$ must be done numerically:

```
dist = NIntegrate[120 (1 - e^(-10 t^2)), {t, 0, 2}]
```

```
206.37005
```

The car travels 206.370 kilometers during the first two hours.

b)

We must find the value of $\frac{d}{dt} g[x(t)]$ when $t = 2$.

```
g[x_] = 0.05 x (1 - e^(-x/2))
```

```
0.05 (1 - e^(-x/2)) x
```

```
PreliminaryExpression = D[g[x[t]], t] /. t → 2
```

```
0.05 (1 - e^(-x[2]/2)) x′[2] + 0.025 e^(-x[2]/2) x[2] x′[2]
```

Now we substitute appropriate values for $x(2)$ and $x'(2)$. We obtained $x(2)$ in part a), and we are given $x'(t) = r(t)$.

```
(PreliminaryExpression /. {x[2] → dist, x'[2] → (120 (1 - e^(-10 t^2)) /. t → 2)})
```

```
6.
```

The rate of change with respect to time of the number of liters of gasoline used by the car when $t = 2$ hours is 6 liters/hour.

c)

We must solve $120\left(1 - e^{-10 t^2}\right) = 80$ in order to find the time at which the car reaches 80 kilometers per hour:

```
Solutions = Solve[120 (1 - e^(-10 t^2)) == 80.0, t]
```

```
Solve::ifun : Inverse functions are being used by Solve, so some
    solutions may not be found; use Reduce for complete solution information. >>
```

```
{{t → -0.33145321}, {t → 0.33145321}}
```

The error message arises because *Mathematica* is thinking about complex solutions; we want real solutions, and *Mathematica* has found all of them. We are interested only in $t > 0$, so we ignore the negative solution. We extract the positive solution:

```
b = t /. Solutions[[2]]
```

```
0.33145321
```

When we have reached a speed of 80 kilometers per hour, the distance traveled is (in kilometers) is given by an integral, as in part a):

```
dist80 = NIntegrate[120 (1 - e^(-10 t^2)), {t, 0, b}]
```

```
10.794097
```

And the amount of fuel consumed up to that time is

```
g[dist80]
```

```
0.53726
```

We've consumed 0.537 liters of fuel at the moment speed reaches 80 kilometers/hour.

Problem 3

a)

We'll store the table and define a function named **RiverDepth** taking on appropriate values:

```
T = {{0, 0}, {8, 7}, {14, 8}, {22, 2}, {24, 0}}
```

```
{{0, 0}, {8, 7}, {14, 8}, {22, 2}, {24, 0}}
```

```
Map[(RiverDepth[#[[1]]] = #[[2]]) &, T]
```

```
{0, 7, 8, 2, 0}
```

And here is the trapezoidal approximation of $\int_0^{24} \text{RiverDepth}(x)\, dx$:

```
((RiverDepth[0] + RiverDepth[8]) (8 - 0) + (RiverDepth[8] + RiverDepth[14]) (14 - 8) +
   (RiverDepth[14] + RiverDepth[22]) (22 - 14) + (RiverDepth[22] + RiverDepth[24]) (24 - 22)) / 2
```

```
115
```

The area of the cross-section is approximately 115 square feet.

b)

We integrate area times volumetric flow with respect to time and divide by the length of the time interval to obtain the average volumetric flow:

$$\frac{115}{120} \int_0^{120} \left(16 + 2\,\mathrm{Sin}\left[\sqrt{t + 10}\right]\right) dt$$

$$\frac{23}{6} \left(480 + \sqrt{10}\,\mathrm{Cos}\left[\sqrt{10}\right] - \sqrt{130}\,\mathrm{Cos}\left[\sqrt{130}\right] - \mathrm{Sin}\left[\sqrt{10}\right] + \mathrm{Sin}\left[\sqrt{130}\right]\right)$$

Numerically, this is

```
N[%]
```

```
1807.1697
```

Average volumetric flow from $t = 0$ to $t = 120$ is 1807.170 cubic feet per minute.

c)

Once again, we integrate depth from 0 to 24:

$$\int_0^{24} 8.0\,\mathrm{Sin}\left[\frac{\pi\, x}{24}\right] dx$$

```
122.231
```

Based on this model, the area of the cross-section is 122.231 square feet.

d)

We must again integrate area times volumetric flow, this time using the area given by the model, and now with t varying over the interval from 40 to 60:

$$\frac{\int_0^{24} 8.0\,\text{Sin}\left[\frac{\pi x}{24}\right] dx}{60 - 40} \int_{40}^{60} \left(16 + 2\,\text{Sin}\left[\sqrt{t + 10}\right]\right) dt$$

2181.9126

The average volumetric flow during the interval $40 \leq t \leq 60$ is 2181.913 cubic feet per minute. This value exceeds the given safety limit of 2100 cubic feet per minute, and indicates that water must be diverted.

Part B

Problem 4

a)

By the Fundamental Theorem of Calculus and the Chain Rule, $f'(x) = \frac{d}{dx} \int_0^{3x} \sqrt{4 + t^2}\, dt = 3\sqrt{4 + 9x^2}$, and $g'(x) = f'(\sin x) \cos x = 3 \cos x \sqrt{4 + 9 \sin^2 x}$.

b)

The slope of the tangent line to $y = g(x)$ at $x = \pi$ is $g'(\pi)$. By part a) of this problem, $g'(\pi) = 3 \cos \pi \sqrt{4 + 9 \sin^2 \pi} = -6$. An equation for the tangent line to the curve $y = g(x)$ at the point corresponding to $x = \pi$ is therefore $y = g(\pi) + g'(\pi)(x - \pi) = 0 - 6(x - \pi))$, or $y = 6(\pi - x)$.

c)

When $x > 0$, the value $f(x)$ is defined as the integral of a positive quantity over an interval $[0, 3x]$, and from this it follows that f is an increasing function throughout the interval $[0, \infty)$. But when we restrict the sine function to $[0, \pi]$, the range is a subset of $[0, \infty)$, and consequently, the maximum value of $g(x) = f(\sin x)$ on $[0, \pi]$ occurs at the value of x which maximizes $\sin x$ on $[0, \pi]$, that is, at $x = \pi/2$. But $g(x) = f(\sin x) = \int_0^{3 \sin x} \sqrt{4 + t^2}\, dt$, and we conclude that the maximal value taken on by g in the interval $[0, \pi]$ is $g(\pi/2) = \int_0^3 \sqrt{4 + t^2}\, dt$.

Note: For the curious, this value is

$$\int_0^3 \sqrt{4 + t^2}\, dt$$

$$\frac{3\sqrt{13}}{2} + 2\,\text{ArcSinh}\left[\frac{3}{2}\right]$$

Problem 5

a)

Inflection points occur at local extrema of g'. There are two such on the given graph: One at $x = 1$, and one at $x = 4$.

b)

From the picture, we see that $g'(x) < 0$ throughout the intervals $[-3, -1)$ and $(2, 6)$, while $g'(x) > 0$ throughout the intervals $(-1, 2)$ and $(6, 7]$. Consequently g is decreasing on the intervals $[-3, -1\}$ and $[2, 6]$, and is increasing on the intervals $[-1, 2]$ and $[6, 7]$. It follows that the absolute maximum value of $g(x)$ must lie either at $x = 2$ where we pass from an interval where g increases to an interval where g decreases, or at one of the endpoints of the interval $[-3, 7]$.

We are given $g(2) = 5$. Making repeated use of the fact that the area of a triangle is one-half its altitude times its base, and that the Fundamental Theorem of Calculus guarantees us that $g(x) = g(2) + \int_2^x g'(t)\,dt$, we find that $g(-3) = 5 - \left(\frac{3}{2} - 4\right) = \frac{15}{2}$ and $g(7) = g(2) + \int_2^7 g'(t)\,dt = 5 - 4 + \frac{1}{2} = \frac{3}{2}$. We now seee that $g(-3) = \frac{15}{2}$ gives the absolute maximum value for $g(x)$ when $-3 \le x \le 7$.

c)

The average rate of change of $g(x)$ on $[-3, 7]$ is $[g(7) - g(-3)]/[7 - (-3)]$. In part b), above, we found that $g(-3) = 15/2$; $g(7) = 3/2$. Thus, the average rate of change of g on $[-3, 7]$ is $(3/2 - 15/2)/(7+3) = -3/5$.

d)

The average rate of change of $g'(x)$ on $[-3, 7]$ is $[g'(7) - g'(-3)]/[7 - (-3)]$. From the graph, we have $g'(-3) = -4$; $g'(7) = 1$. Thus, the average rate of change of g' on $[-3, 7]$ is $[1 - (-4)]/(7 + 3) = 1/2$.

The Mean Value Theorem does not apply to the function g' on the interval $[-3, 7]$, because the hypotheses of that theorem require that $g''(x)$ exist for all values of $x \in (-3, 7)$. However $g''(-1)$, $g''(1)$, and $g''(4)$ do not exist for this function.

Problem 6

a)

If $x^2 + 2x + y^4 + 4y = 5$, then, treating y as an implicitly defined function of x and differentiating both sides of the equation with respect to x, we obtain $2x + 2 + 4y^3 y' + 4y' = 0$. Then $\left(4y^3 + 4\right) y' = -2x - 2$, or $y' = -(x + 1)/\left[2\left(y^3 + 1\right)\right]$, as desired.

b)

At $(-2,\ 1)$, we have $y' = -(-2+1)/\left[2\left(1^3+1\right)\right] = 1/4$. An equation for the tangent line is therefore $y = 1 + \frac{1}{4}(x+2)$.

c)

We begin anew, with $x^2 + 2x + y^4 + 4y = 5$, and treating x as an implicitly defined function of y. Differentiating both sides with respect to y, we obtain $2xx' + 2x' + 4y^3 + 4 = 0$, whence $\frac{dx}{dy} = -2\frac{(1+y^3)}{1+x}$. This vanishes, giving us a vertical tangent line, at any point on the curve where $y = -1$. The corresponding values of x are then given by $x^2 + 2x + (-1)^4 + 4\cdot(-1) = 5$, or, equivalently, $x^2 + 2x - 8 = 0$. We conclude that the curve has vertical tangent lines at the points $(-4,\ -1)$ and $(2,\ -1)$.

d)

At any point where this curve meets the x-axis, we must have $y = 0$, whence $x^2 + 2x + 0^4 + 4\cdot 0 = 5$. Thus, at such a point we must have $x^2 + 2x + 1 = 6$, or $(x+1)^2 = 6$. Therefore, $y = 0$ implies that $x = -1 \pm \sqrt{6}$, or that $x = \sqrt{6} - 1$ or $x = -\sqrt{6} - 1$. But from our expression for y' in part a), $x = \sqrt{6} - 1$ and $y = 0$ together imply that $y' = -\sqrt{6}\big/2$, while $x = -\sqrt{6} - 1$ and $y = 0$ together imply that $y' = \sqrt{6}\big/2$. If follows that this curve can't have a horizontal tangent at any of its x-intercepts.

Note

The test writers have done the readers a grave disservice with part c) of this problem. The implicit differentiation technique depends for its justification upon the Implicit Function Theorem, which we can't expect students in AB Calculus to know. Among the hypotheses of that theorem is that, when we want to deal with a function $y(x)$ defined implicitly in some neighborhood of a point (x_0, y_0) by an equation of the form $F(x,\ y) = 0$, we must have $F_y(x_0, y_0) \neq 0$ before we may draw any conclusions about the existence (let alone differentiability) of $y(x)$ at x_0. This means that *it is not technically correct to conclude that a curve $F(x,\ y) = 0$ has a vertical tangent by examining a y' that we have obtained by implicit differentiation*. For a deeper discussion of this difficulty, including a counter-example, see my note at

<http://clem.mscd.edu/~talmanl/PDFs/APCalculus/OnImplicitDifferentiation.pdf>.