Solutions to the 2009 AP Calculus AB Free Response Questions

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Problem 1.

■ a.

At time t = 7.5, the acceleration of Caren's bicycle is $-\frac{1}{10}$ miles per minute per minute.

■ b.

The integral $\int_0^{12} |\, v(t) \,|\, dt$ gives, in miles, the total distance that Caren traveled during the period $0 \leq t \leq 12$. The value of this integral is $\frac{9}{5}$ miles.

■ c.

Her turn-around time corresponds to the point on the graph where the sign of her velocity changes from positive to negative. That's $t = 2$ minutes.

■ d.

Caren lives $\int_5^{12} v(t)\, dt = \frac{7}{5}$ miles from school because she left home at $t = 5$, arrived at school at $t = 12$, traveled in one direction only, and the distance she traveled during that time is given by the integral. Larry's distance is the integral of his velocity over the interval [0, 12], or

$$\frac{\pi}{15}\int_0^{12}\mathrm{Sin}\left[\frac{\pi}{12}\,t\right]\,dt$$

$$\frac{8}{5}$$

At $\frac{8}{5}$ miles, Larry lives farther than Caren, at $\frac{7}{5}$ miles from the school. Thus, Caren lives closer.

Problem 2.

▪ a.

At time $t = 2$, the auditorium contains $\int_0^2 (1380\,t^2 - 675\,t^3)\,dt$ people.

$$\int_0^2 (1380\,t^2 - 675\,t^3)\,dt$$

980

That's 980 people.

▪ b.

We are given $R(t) = 1380\,t^2 - 675\,t^3$, so that $R'(t) = 2760\,t - 2025\,t^2 = 15\,t\,(184 - 135\,t)$. Thus, $R(t)$ is increasing on the interval $[0, \frac{184}{135}]$ and decreasing on the interval $[\frac{184}{135}, 2]$, because $R'(t)$ is positive on the first of these intervals and negative on the second. It follows that the maximal rate at which people enter the auditorium is at $t = \frac{184}{135}$ hours.

▪ c.

We have

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R[t_] = 1380 t^2 - 675 t^3
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$1380\,t^2 - 675\,t^3$

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Dw[t_] = (2 - t) R[t]
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$(2 - t)\,(1380\,t^2 - 675\,t^3)$

By the Fundamental Theorem of Calculus, the difference $w(2) - w(1)$ is given by

$$\int_1^2 \mathrm{Dw}[\tau]\,d\tau$$

$$\frac{775}{2}$$

Total wait time for those who enter the auditorium after $t = 1$ is $\frac{775}{2}$ hours.

■ d.

From part (a) of this problem, above, we know that there are 980 people in the auditorium at time $t = 2$. We also know that the total wait time for these 980 people is

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∫₀² Dw[τ] ⅆτ
760
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760 hours. Consequently, average waiting time is $\frac{760}{980} = \frac{38}{49}$ hours.

Problem 3.

■ a.

Mighty's profit on a cable of length k meters is given by $P(k) = 120\,k - \int_0^k 6\sqrt{x}\,dx$, in dollars.

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P[k_] = 120 k - 6 ∫₀ᵏ √x ⅆx
120 k - 4 k^(3/2)
P[25]
2500
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Thus, profit on a 25-meter cable is $2500

■ b.

The integral $\int_{25}^{30} 6\sqrt{x}\,dx$ gives the cost, in dollars, to Mighty for building the last five meters of a 30-meter cable.

■ c.

As we saw in part (a) of this problem, above, Mighty's profit, in dollars, on the sale of a cable that is k meters long is $120\,k - 6\int_0^k \sqrt{x}\,dx$, or $120\,k - 4\,k^{3/2} = 4\,k\left(30 - \sqrt{k}\right)$.

■ d.

We first observe that profit for a cable k meters long, which is given by $4k(30 - \sqrt{k})$, is non-negative only when $0 \le k \le 900$ and is zero at the endpoints of this interval. Thus, the maximum does not occur at an endpoint of the interval (or outside the interval). However, the profit function is continuous, so there must be a maximum ocurring at some $k \in [0, 900]$. As we have observed, we must actually have $k \in (0, 900)$, so k must be a critical number for the profit function. Noting that the derivative

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P'[k]
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$$120 - 6\sqrt{k}$$

of the profit function vanishes only when $k = 400$, we conclude that the maximum profit must come at $k = 400$. The maximal profit is therefore $P(400) = 16,000$ dollars.

We also notice that if cables longer than 900 meters ever become very popular, Mighty should restructure something.

Problem 4.

■ a.

The desired area is

$$\int_0^2 (2x - x^2)\, dx$$

$$\frac{4}{3}$$

■ b.

The desired volume is

$$\int_0^2 \text{Sin}\left[\frac{\pi}{2} x\right] dx$$

$$\frac{4}{\pi}$$

■ c.

Solving both equations for y in terms of x gives us $x = \frac{1}{2} y$ for the first and $x = \sqrt{y}$ for the second. The length of the base of the square corresponding to $y = y_0$ is therefore $\sqrt{y_0} - \frac{1}{2} y_0$, and the desired volume is therefore $\int_0^4 \left(\sqrt{y} - \frac{1}{2} y\right)^2 dy$. (For those who are unable to comply with the instruction no to evaluate the integral,, this is $\frac{8}{15}$.)

Problem 5.

■ a.

$f'(4) \sim \frac{f(5)-f(3)}{5-3} = \frac{-2-4}{5-3} = \frac{-6}{2} = -3.$

■ b.

$\int_2^{13} (3 - 5\, f'(x))\, dx$.

$= [3\,x - 5\, f(x)]\,|_2^{13}$

$= [3 \cdot 13 - 5 \cdot f(13)] - [3 \cdot 2 - 5 \cdot f(2)]$

$= (39 - 30) - (6 - 5) = 8$

■ c.

The desired left Riemann sum is $f(2)\,(3-2) + f(3)\,(5-3) + f(5)\,(8-5) + f(8)\,(13-8) = 1 + 8 - 6 + 15 = 18.$

■ d.

An equation for the line tangent to the curve $y = f(x)$ at $x = 5$ is $y = f(5) + f'(5)\,(x - 5)$, or $y = -2 + 3\,(x - 5)$. Now $f''(x) < 0$ for all x in the interval $[5, 8]$, so the curve is concave downward throughout that interval; thus, the tangent line at $x = 5$ lies above the curve throughout the interval $[5, 8]$. That is, when $5 \le x \le 8$, $f(x) \le -2 + 3\,(x - 5)$. Consequently, $f(7) \le -2 + 3\,(7 - 5) = 4$.

On the other hand, $f''(x) < 0$ on $[5, 8]$ implies that the curve $y = f(x)$,being concave downward, lies above the secant line determined by the points $(5, f(5)) = (5, -2)$ and $(8, f(8)) = (8, 3)$. An equation for this secant line is $y = f(5) + \frac{f(8)-f(5)}{8-5}\,(x - 5) = -2 + \frac{5}{3}\,(x - 5)$. Consequently, when $5 \le x \le 8$, we have $-2 + \frac{5}{3}\,(x - 5) \le f(x)$. Thus, $\frac{4}{3} \le -2 + \frac{5}{3}\,(7 - 5) \le f(7)$.

Problem 6.

■ a.

The graph of f has an inflection point where the graph of f' has a local extreme, because at such points f'' must undergo a sign change. From what we are given about the graph, we see that f' has local extremes at $x = -2$ and at $x = 0$. Therefore, f has an inflection point at $x = -2$ and another at $x = 0$.

■ b.

For any x in $[-4, 4]$, we have $f(x) = f(0) + \int_0^x f'(\xi)\, d\xi$, by the Fundamental Theorem of Calculus. We are given that $g(x)$ has a semicircle for its graph over $-4 \le x \le 0$, so $f(0) + \int_0^{-4} g(\xi)\ \ d\xi = 5 - (8 - \frac{1}{2}\pi \cdot 2^2) = 5 - (8 - 2\pi) = -3 + 2\pi = f(-4)$. (Here we have evaluated the integral by subtracting the area of a semicircle of radius 2 from a rectangle of height 2 and width 4.)

On the other hand, when $x \ge 0$, we are given $f'(x) = 5\, e^{-x/3} - 3$, so $f(4) = f(0) + \int_0^4 f'(\xi)\, d\xi = 5 + \int_0^4 (5\, e^{-\xi/3} - 3)\, d\xi = 8 - 15\, e^{-4/3}$.

■ c.

From what we are given about the graph of f', we know that $f'(x) \ge 0$ for all x in the interval $[-4,\ 3\ln(5/3)]$, and is actually positive for all but a finite number (2 such points, in fact) of points in that interval. Consequently, f is an increasing function on $[-4,\ 3\ln(5/3)]$. We also know that $f'(x) \le 0$ when $x \in [3\ln(5/3), 4]$, with finitely many zeros (1, in fact) in that interval. Thus, f is decreasing on the interval $[3\ln(5/3),\ 4]$. From this it follows that f has an absolute maximum for $-4 \le x \le 4$ at $x = 3\ln(5/3)$.