# Solutions to the (Form B)
# 2009 AP Calculus AB
# Free Response Questions

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## Problem 1.

### ▪ a.

At time $t$, the radius of the tree, $R(t)$, in centimeters, is given by $R(t) = 6 + \frac{1}{16}\int_0^t (3 + \text{Sin}[\tau^2])\,d\tau$,

$$6 + \frac{1}{16}\,\mathbf{NIntegrate}[\mathbf{3} + \mathbf{Sin}[\tau^2],\ \{\tau,\ \mathbf{0},\ \mathbf{3}\}]$$

```
6.6108477
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When $t = 3$, the radius of the tree is about 6.611 cm.

### ▪ b.

We have $A(t) = \pi\,[R(t)]^2$, whence, by implicit differentiation, $A'(t) = 2\pi\,R(t)\,R'(t)$. Thus, $A'(3) = 2\pi\,R(3)\,R'(3) = 2\pi \cdot 6.6108477 \cdot \frac{1}{16}[3 + \sin(9)]$

$$\mathbf{2}\,\pi\,(\mathbf{6.6108477})\,\frac{1}{16}\,(\mathbf{3} + \mathbf{Sin}[\mathbf{9}])$$

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8.8581115
```

The area is then increasing at the rate of 8.858 square centimeters per year.

### ■ c.

The integral $\int_0^3 A'(t)\,dt = A(3) - A(0) = \pi\,[(6.6108477)^2 - 36]$ represents the change, in square centimeters, in the area of the tree's cross-section at the given height over the time period $0 \le t \le 3$.

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π ((6.6108477)^2 - 36)
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24.200654
```

The area of the cross-section is 24.201 square centimeters larger when $t = 3$ than it was when $t = 0$.

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## Problem 2.

### ■ a.

Let $D(t)$ denote the distance, in meters, from the road to the edge of the water at time $t$ hours after the beginning of the storm. We are given $D(0) = 35$, $D'(t) = \sqrt{t} + \cos t - 3$. By the Fundamental Theorem of Calculus, $D(t) = 35 + \int_0^t \left(\sqrt{\tau} + \cos\tau - 3\right) d\tau = 35 - 3\,t + \frac{2}{3}\,t^{3/2} + \sin t$.

```
35 - 3 t + (2/3) t^(3/2) + Sin[t] /. t → 5.0
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26.494636
```

Consequently, at the end of the 5-hour storm, we have $D(5) = 26.495$ meters.

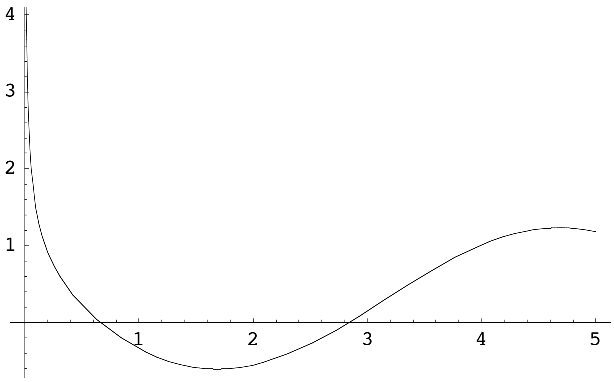
### ■ b.

If $f'(4) = 1.007$, then $D''(4) = 1.007$, so after four hours of the storm, the rate at which distance from road to water is changing is increasing at 1.007 meters per hour per hour.

### ■ c.

We are to find the absolute minimum of $f(t)$ on the interval [0, 5]. Such a minimum lies at a critical point or an end-point. The critical points for $f$ are the zeros of $f'(t) = \frac{1}{2\sqrt{t}} - \sin t$.

$$\mathbf{Plot}\left[\frac{\mathbf{1}}{\mathbf{2}\sqrt{\mathbf{t}}} - \mathbf{Sin[t],\ \{t,\ 0,\ 5\}}\right]$$



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- Graphics -
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$$\mathbf{FindRoot}\left[\frac{\mathbf{1}}{\mathbf{2}\sqrt{\mathbf{t}}} - \mathbf{Sin[t],\ \{t,\ 1.0\}}\right]$$

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{t → 0.66186548}
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$$\mathbf{FindRoot}\left[\frac{\mathbf{1}}{\mathbf{2}\sqrt{\mathbf{t}}} - \mathbf{Sin[t],\ \{t,\ 3.0\}}\right]$$

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{t → 2.8403832}
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$$\mathbf{f[t\_]\ =\ \sqrt{t}\ +Cos[t]\ -\ 3}$$

$$-3+\sqrt{t}+\mathrm{Cos}[t]$$

The values of $f$ at the end-points and critical points are

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Map[f, {0, 0.66186548, 2.8403832, 5.0}]
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{-2, -1.3976017, -2.2696347, -0.48026984}
```

The smallest of these is $-2.2696347$, so the distance from water to road was decreasing most rapidly about 2.840 hours after the storm began.

## ■ d.

If sand is restored to the beach in such a way that the rate of change of the distance from water to road is $g(p)$ meters per day, where $p$ is the number of days since pumping began, then, by the Fundamental Theorem of Calculus, $P$, the number of days of pumping required to restore the original distance between road and water, satisfies (approximately) the equation $35 = 26.495 + \int_0^P g(p)\,dp$. For an exact equation, we can replace 26.495 with $35 + \int_0^5 \left(\sqrt{\tau} + \cos\tau - 3\right) d\tau$.

## Problem 3.

We don't appear to have been given quite enough information to solve this problem. We must assume that the line segment and the curved portion of the curve meet at the point (0, 2). In what follows, we will make this assumption.

### ■ a.

The line segment that gives the portion of the curve that lies to the left of the $y$-axis has slope very close to $\frac{2}{3}$, so $\lim_{h\to 0^-} \frac{f(h)-f(0)}{h} \sim \frac{2}{3}$, while it is apparent from the graph that $\lim_{h\to 0^+} \frac{f(h)-f(0)}{h} < 0$. Consequently the left-hand and right-hand limits of the difference quotient do not agree; $f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h}$ cannot exist, and $f$ is not differentiable at $x = 0$.

### ■ b.

The average rate of change of $f$ over the interval $[a, 6]$ is $\frac{f(a) - f(6\}}{a-6}$. This can be zero only if $f(a) = f(6) = 1$ while $a \neq 6$. The horizontal line through $(6, f(a)) = (6, 1)$ intersects the curve in just two other points, so there are just two values of $a$ for which the average rate of change of $f$ over $[a, 6]$ is zero.

### ■ c.

We note that $f$ is continuous on $[3, 6]$ and differentiable on $(3, 6)$, so the Mean Value guarabntees that there is a number $c \in [3, 6]$ such that $f'(c) = \frac{f(6)-f(3)}{6-3} = \frac{1}{3}$. Thus, we may take $a = 3$.

### ■ d.

If $g(x) = \int_0^x f(t)\,dt$, then $g'(x) = f(x)$, and $g''(x) = f'(x)$. Thus $g$ is concave upward on intervals where $f'(x) > 0$, or on intervals where $f$ is increasing. We conclude that $g$ is concave upward on $(-4, 0)$ and on $(3, 6)$. Whether or not we may conclude that $g$ is concave upward on $[-4, 0]$ or on $[3, 6]$ depends upon which of several definitions of concavity we choose.

## Problem 4.

### ■ a.

The area of $R$ is $\int_0^4 \left(\sqrt{x} - \frac{x}{2}\right) dx$.

$$\int_0^4 \left(\sqrt{\mathbf{x}} - \frac{\mathbf{x}}{\mathbf{2}}\right) d\mathbf{x}$$

$$\frac{4}{3}$$

### ■ b.

The volume of the solid described is $\int_0^4 \left(\sqrt{x} - \frac{x}{2}\right)^2 dx$.

$$\int_0^4 \left(\sqrt{\mathbf{x}} - \frac{\mathbf{x}}{2}\right)^2 d\mathbf{x}$$

$$\frac{8}{15}$$

### ■ c.

The volume generated by revolving $R$ about the line $y = 2$ is $\pi \int_0^4 \left[(2 - \frac{x}{2})^2 - \left(2 - \sqrt{x}\right)^2\right] dx = \pi \int_0^4 \left(4\sqrt{x} - 3x + \frac{x^2}{4}\right) dx$. Alternately, we may write the volume as $2\pi \int_0^2 (2y - y^2)(2 - y)\, dy = 2\pi \int_0^2 (4y - 4y^2 + y^3)\, dy$.

Note: Those who weren't able to comply with the instruction not to evaluate the integral should have found that the volume in question is $8\pi/3$.

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## Problem 5.

### ■ a.

If $g(x) = \exp[f(x)]$, then $g'(x) = f'(x)\exp[f(x)]$, so $g'(1) = f'(1)\exp[f(1)] = -4e^2$. Hence, an equation for the line tangent to $y = g(x)$ at $x = 1$ is $y = g(1) + g'(1)(x - 1)$, or $y = e^2 - 4e^2(x - 1)$.

### ■ b.

By the First Derivative Test, $g$ has a local maximum at any point where $g'(x)$ changes sign from positive to negative. But $g'(x) = f'(x)\exp[f(x)]$, and $\exp[f(x)]$ is always positive. Therefore the local maxima of $g$ are to be found at points where $f'(x)$ changes sign from positive to negative. From the graph given, we see that there is just one such point: $x = -1$. The function $g$ therefore has a local maximum only at $x = -1$ in the interval $(-1.2,\ 3.2)$.

### ■ c.

Because $g''(x) = \exp[f(x)]\,[(f'(x))^2 + f''(x)]$ and $\exp[f(x)] > 0$, the sign of $g''(x)$ is the same as the sign of $[(f'(x))^2 + f''(x)]$. Now, as is given, $(f'(-1))^2 = 0$, and we see from the graph that $f'$ being a decreasing function in a neighborhood of $x = -1$ it must be the case that $f''(-1) < 0$. Consequently, $g''(-1) < 0$.

### ■ d.

The average rate of change of $g'$ over the inverval [1, 3] is $\frac{g'(3) - g'(1)}{3 - 1}$. Now, $g'(1) = -4\,e^2$, as we saw in part (a) of this problem, above. We also have $g'(3) = f'(3)\exp[f(3)] = 0$. ($f'(3) = 0$ is given.) The desired average rate of change is therefore $\frac{0 - (-4\,e^2)}{2} = 2\,e^2$.

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## Problem 6.

### ■ a.

Acceleration at time $t = 36$ is approximately $\frac{v(36+4) - v(36-4)}{(36+4) - (36-4)} = \frac{7 - (-4)}{8} = \frac{11}{8}$ meters per second per second.

### ■ b.

The quantity $\int_{20}^{40} v(t)\,dt$ gives the difference $x(40) - x(20)$ between the particle's position when $t = 40$ and its position when $t = 20$. This difference is approximately
$\frac{1}{2}[[v(25) + v(20)](25 - 20) + [v(32) + v(25)](32 - 25) + [v(40) + v(32)](40 - 32)]$,
which is $\frac{1}{2}[(-18)\cdot 5 + (-12)\cdot 7 + (3)\cdot 8] = -75$ meters.

### ■ c.

The particle must change direction somewhere in the interval [8, 20] because the velocity function $v(t)$ takes on a positive value when $t = 8$ and a negative value when $t = 20$. It must also change direction somewhere in the interval [32, 40] because $v(32) = -4$ and $v(40) = 7$.

### ■ d.

By the Fundamental Theorem of Calculus, $x(t) = x(0) + \int_0^t v(\tau)\,dt = 7 + \int_0^t v(\tau)\,d\tau$. But if acceleration, which is $v'$, is positive on (0, 8) and $v(0) = 3$, then $v(\tau) \geq 3$ for all $\tau \in [0, 8]$. Thus $x(8) = 7 + \int_0^8 v(\tau)\,d\tau \geq 7 + \int_0^8 3\,d\tau = 31 > 30$.

**Alternate Solution:** Because velocity is defined everywhere we need to think about it, the position function is always continuous. We can apply the Mean Value Theorem: Assume, by way of contradiction, that $x(8) < 30$. There must be $t_0 \in (0, 8)$ such that $v(t_0) = [x(8) - x(0)]/(8 - 0) < 23/8 < 3$. And then there must also be $t_1 \in (0, t_0)$ such that $v'(t_1) = [v(t_0) - v(0)]/(t_0 - 0) = [v(t_0) - 3]/t_0 < 0$. But $v'(t_1)$ is acceleration at $t_1$, and $t_1$ lies in $(0, t_0) \subseteq (0, 8)$. This contradicts what was given: Acceleration is positive on (0, 8).