Solutions to the
1998 AP Calculus BC Exam
Free Response Questions

Louis A. Talman
Department of Mathematical & Computer Sciences
Metropolitan State College of Denver

Problem 1.

a.

The area of the region *R* is

$$\int_0^4 \left(8 - x^{3/2}\right) dx$$

$$\frac{96}{5}$$

b.

The volume of the solid generated by revolving *R* about the *x*-axis is

$$\pi \int_0^4 \left(8 - x^{3/2}\right)^2 dx$$

$$\frac{576\,\pi}{5}$$

c.

The value of k is given by

```
FindRoot[π ∫_0^k (8 - x^{3/2})^2 ⅆx == π/2 ∫_0^4 (8 - x^{3/2})^2 ⅆx, {k, 1}]

{k → 0.99490352235}
```

$k = 0.995$, to three digits beyond the decimal.

Problem 2

a.

$\lim_{x\to-\infty} 2\,x\,e^{2x} = \lim_{x\to-\infty} \frac{2x}{e^{-2x}}$. L'Hôpital's Rule is applicable to the latter expression. Thus, $\lim_{x\to-\infty} \frac{2x}{e^{-2x}} = \lim_{x\to-\infty} \frac{2}{-2\,e^{-2x}} = 0$.

b.

If $f[x] = 2\,x\,e^{2x}$, then $f'[x] = (2 + 4\,x)\,e^{2x}$. Consequently, $f'[x] = 0$ only when $x = -\frac{1}{2}$. Now (by part a, above) $\lim_{x\to-\infty} f[x] = 0$, while $\lim_{x\to\infty} f[x] = \infty$. Consequently there are numbers x_1 and x_2, $x_1 < \frac{-1}{2} < x_2$, such that $x \le x_1$ implies that $f[x] \ge -\frac{1}{100} > f[-1/2] = -e^{-1}$ and $x_2 \le x$ implies that $f[x] \ge -\frac{1}{100} > f[-1/2]$. But f must have an absolute minimum in the interva l $[x_1, x_2]$, and it cannot be located at either x_1 or x_2. Because $x = -1/2$ is the only critical point in this interval, it must give the absolute minimum for $f[x]$ when $x_1 \le x \le x_2$, and therefore for $-\infty < x < \infty$.

c.

By the observations we have made in part b. above, the range of f is $\left[-e^{-1}, \infty\right)$.

d.

Let us assume, for the moment, that $b > 0$. Then, arguing as we have in parts a. and b. above, we find that $f[x] = b\,x\,e^{b\,x}$ has an absolute minimum at $x = -\frac{1}{b}$. This minimum value is $f\left[\frac{-1}{b}\right] = -e^{-1}$, which is independent of b. If $b < 0$, we obtain the same result after the change of variables $u = -x$, which amounts to a reflection about the y-axis.

Problem 3.

■ a.

The third-degree Taylor polynomial for f about $x = 0$ is $f[0] + f'[0]\,x + \frac{f''[0]}{2}\,x^2 + \frac{f'''[0]}{6}\,x^3$ or $5 - 3\,x + \frac{1}{2}\,x^2 + \frac{2}{3}\,x^3$. Thus, $f[0.2]$ is approximately

```
5 - 3 x + 1/2 x^2 + 2/3 x^3 /. x → 0.2

4.42533333333
```

■ b.

We can obtain the third-degree Taylor polynomial for $g[x] = f\left[x^2\right]$ about $x = 0$ by substituting x^2 for x in the Taylor polynomial for f and then truncating. This gives $5 - 3\,x^2 + \frac{1}{2}\,x^4$.

■ c.

We can obtain the third-degree Taylor polynomial for $h[x] = \int_0^x f[t]\,dt$ by integrating that of f term by term and truncating. We obtain $5\,x - \frac{3}{2}\,x^2 + \frac{1}{6}\,x^3$.

■ d.

We cannot determine $h[1] = \int_0^1 f[t]\,dt$ from what is given. It is possible that $f[x] = 5 - 3\,x + \frac{1}{2}\,x^2 + \frac{2}{3}\,x^3$, in which case we would have $h[t]$ given by

$$\int_0^1 \left(5 - 3\,t + \frac{1}{2}\,t^2 + \frac{2}{3}\,t^3\right) dt$$

$$\frac{23}{6}$$

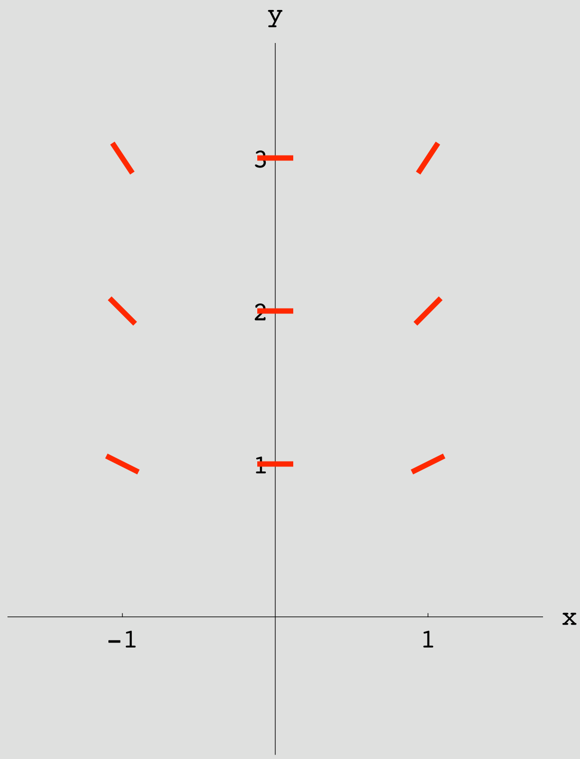
However, it is also consistent with what has been given that $f[x] = 5 - 3\,x + \frac{1}{2}\,x^2 + \frac{2}{3}\,x^3 + x^4$, and if this is the case, then $h[1]$ would be given by

$$\int_0^1 \left(5 - 3\,t + \frac{1}{2}\,t^2 + \frac{2}{3}\,t^3 + t^4\right) dx$$

$$5 - 3\,t + \frac{t^2}{2} + \frac{2\,t^3}{3} + t^4$$

Problem 4

a.



b.

Euler's Method is given by

```
F[x_, y_] = x y / 2
```

$\frac{x\,y}{2}$

```
EulerStep[{x_, y_}, h_] = {x + h, y + F[x, y] h}
```

$\left\{h+x,\ y+\frac{h\,x\,y}{2}\right\}$

The first Euler step with $h = 0.1$ gives

```
EulerStep[{0, 3}, 0.1]
```

```
{0.1, 3}
```

The second gives

```
EulerStep[%, 0.1]
```

```
{0.2, 3.015}
```

Thus, $f[0.2]$ is approximately 3.015.

C.

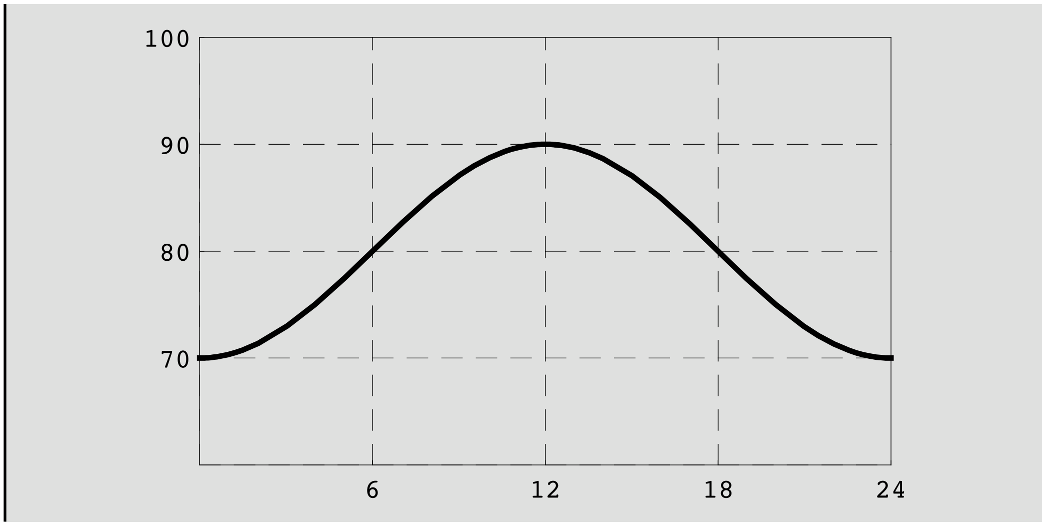
If f is the solution to $\frac{dy}{dx} = \frac{x\,y}{2}$ for which $f[0] = 3$, then $\int_0^x \frac{f'[t]}{f[t]}\,dt = \int_0^x \frac{t}{2}\,dt$, or $\int_{f[0]}^{f[x]} \frac{du}{u} = \int_0^x \frac{t}{2}\,dt$. Hence, $\ln\left(\frac{f[x]}{3}\right) = \frac{x^2}{4}$, and $f[x] = 3\,e^{x^2/4}$. This gives, for $f[0.2]$,

```
3 e^(x^2/4) /. x → 0.2
```

```
3.03015050125
```

Problem 5

a.



b.

Average temperature is $\frac{1}{14-6}\int_6^{14} F[t]\, dt$:

$$\frac{1}{14-6}\int_6^{14}\left(80 - 10\,\mathrm{Cos}\left[\frac{\pi\, t}{12}\right]\right) dt$$

$$\frac{1}{8}\left(640 + \frac{180}{\pi}\right)$$

```
N[%]
```

```
87.1619724391
```

To the nearest degree, this is 87 degrees.

c.

```
FindRoot[80 - 10 Cos[π t/12] == 78, {t, 6}]

{t → 5.23086939781}
```

```
a = t /. %

5.23086939781
```

```
FindRoot[80 - 10 Cos[π t/12] == 78, {t, 18}]

{t → 18.7691306022}
```

```
b = t /. %

18.7691306022
```

The air conditioner ran when $5.231 \le t \le 18.769$.

d.

The approximate total cost is $0.05 \int_a^b \left(2 - 10 \operatorname{Cos}\left[\frac{\pi t}{12}\right]\right) dt$, or

```
0.05 ∫_a^b (2 - 10 Cos[π t/12]) dt

5.09637076636
```

To the nearest cent, this is $5 .10.

Problem 6

a.

If $x'[t] = \frac{1}{\sqrt{2t+1}}$, with $x[0] = -4$, then $\int_0^t x'[\tau]\,dt = \int_0^t \frac{d\tau}{\sqrt{2\tau+1}}$, or $\int_{x[0]}^{x[t]} du = \int_0^t \frac{d\tau}{\sqrt{2\tau+1}}$. Thus, $x[t] + 4 = \sqrt{2t+1} - 1$, so that $x[t] = \sqrt{2t+1} - 5$.

b.

$y[t] = x[t]^3 - 3x[t]$, so $y'[t] = 3x[t]^2 x'[t] - 3x'[t] = \left(3\left(\sqrt{2t+1} - 5\right)^2 - 3\right) \cdot \frac{1}{\sqrt{2t+1}}$.

c.

When $t = 4$, $x[t] = x[4] = \sqrt{2 \cdot 4 + 1} - 5 = -2$ and $y = x^3 - 3x = (-2)^3 - 3(-2) = -8 + 6 = -2$. Also, $x'[4] = \frac{1}{\sqrt{2 \cdot 4+1}} = \frac{1}{3}$, while $y'[t] = \left(3(3-5)^2 - 3\right) \cdot \frac{1}{3} = 3$. Hence, when $t = 4$, speed is $\sqrt{x'[t]^2 + y'[t]^2} = \sqrt{\frac{1}{9} + 9} = \frac{\sqrt{82}}{3}$. At time $t = 4$, the particle is at $(-2, -2)$ with speed $\frac{\sqrt{82}}{3}$.