# Solutions to the
# 1999 AP Calculus BC Exam
# Free Response Questions

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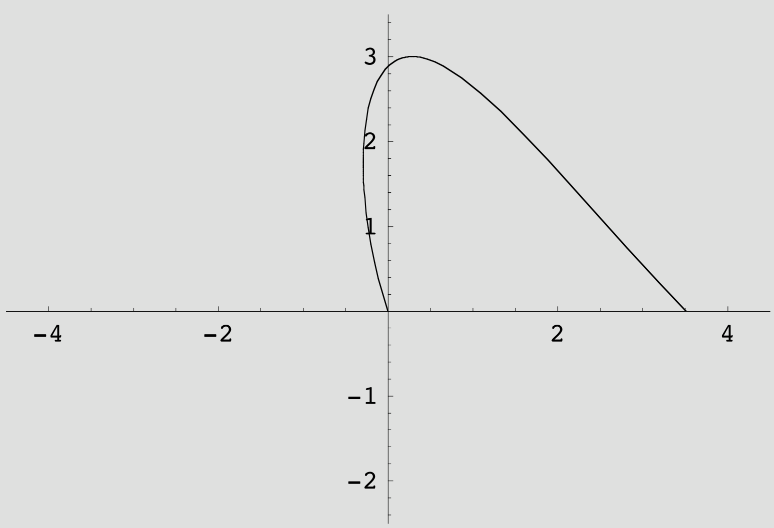
## Problem 1.

### ■ a.

```
ParametricPlot[{t^2/2 - Log[1 + t], 3 Sin[t]},
 {t, 0, π}, AspectRatio → Automatic,
 PlotRange → {{-4.5, 4.5}, {-2.5, 3.5}}]
```




```
- Graphics -
```

When $t = 0$, we have $x = 0$ and $y = 0$, so as $t$ increases from 0 to $\pi$, the curve is traced out from the origin upward to the left, and around back down to its terminal point near $(\frac{7}{2}, 0)$.

## b.

$x[t] = \frac{t^2}{2} - \ln(1+t)$, so $x'[t] = t - \frac{1}{1+t}$. Thus $x'[t] = 0$ when $t - \frac{1}{1+t} = 0$, or when $t^2 + t - 1 = 0$. The only critical point in $(0, \pi)$ is thus $x = \frac{\sqrt{5}-1}{2}$, where $x[t] < 0$. Because $x[0] = 0$, and $x[\pi] > 0$, the minimum occurs at $t = \frac{\sqrt{5}-1}{2}$. At that time we have $x[t]$ given by

```
(t^2 - Log[1 + t] /. t -> (Sqrt[5] - 1)/2) // N

-0.099245814
```

and $y[t]$ given by

```
(3 Sin[t] /. t -> (Sqrt[5] - 1)/2) // N

1.7383018
```

## c.

The particle is on the $y$-axis when $x[t] = 0$. This happens when

```
FindRoot[t^2/2 - Log[1 + t] == 0, {t, 2.0}]

{t → 1.2858887}
```

Let us call this number $T$.

```
T = t /. %[[1]]

1.2858887
```

```
x[t_] = t^2/2 - Log[1 + t]
```

$\frac{t^2}{2} - \text{Log}[1+t]$

```
y[t_] = 3 Sin[t]
```

3 Sin[t]

The speed at time $t = T$ is

```
Sqrt[x'[T]^2 + y'[T]^2]
```

1.1961677

Acceleration at time $t = T$ is

```
{x''[T], y''[T]}
```

{1.191377, -2.8790629}

## Problem 2

### a.

The area of the pictured region is $\int_{-2}^{2}(4 - x^2)\,dx = (4x - \frac{1}{3}x^3)\Big|_{-2}^{2} = \frac{32}{3}$.

### b.

Revolving the pictured region about the $x$-axis produces a solid whos volume is $\pi \int_{-2}^{2}(16 - x^4)\,dx = \frac{256}{5}\pi$.

### c.

$\pi \int_{-2}^{2}\left((k - x^2)^2 - (k - 4)^2\right)dx = \frac{256}{5}\pi$.

For the curious:

```
Solve[∫_{-2}^{2} ((k - x^2)^2 - (k - 4)^2) dx == 256/5, k]

{{k → 24/5}}
```

## Problem 3.

First, let us assign appropriate values to the function *R*:

```
Map[Apply[(R[#1] = #2) &, #] &,
 {{0, 9.6}, {3, 10.4}, {6, 10.8}, {9, 11.2}, {12, 11.4},
  {15, 11.3}, {18, 10.7}, {21, 10.2}, {24, 9.6}}]

{9.6, 10.4, 10.8, 11.2, 11.4, 11.3, 10.7, 10.2, 9.6}
```

### ■ a.

```
Sum[R[k] 6, {k, 3, 21, 6}]

258.6
```

Approximately 258.6 gallons of water flows out of the pipe in the period $0 \le t \le 24$.

### ■ b.

The function *R* is given differentiable on [0, 24], and it is also given that $R[0] = 9.6 = R[24]$. By Rolle's Theorem, there must be a $t \in (0, 24)$ such that $R'[t] = 0$.

### ■ c.

The average rate of flow is approximately $\frac{1}{24} \int_0^{24} \frac{1}{79} (768 + 23\,t - t^2)\, dt = \frac{852}{79}$ gallons per hour.

# Problem 4

## ■ a.

The third-degree Taylor polynomial about $x = 2$ for $f$ is $f[x] = f[0] + f'[0](x-2) + \frac{f''[0]}{2}(x-2)^2 + \frac{f'''[0]}{6}(x-2)^3 = -3 + 5(x-2) + \frac{3}{2}(x-2)^2 - \frac{4}{3}(x-2)^3$.

```
T[x_] = -3 + 5 (x - 2) + 3/2 (x - 2)^2 - 4/3 (x - 2)^3
```

$$-3+5(-2+x)+\frac{3}{2}(-2+x)^2-\frac{4}{3}(-2+x)^3$$

The required approximation for $f[1.5]$ is therefore

```
T[3/2]
```

$$-\frac{119}{24}$$

```
N[%]
```

```
-4.9583333
```

## ■ b.

The Lagrange bound on the error in this third degree Taylor polynomial is $\frac{M}{4!}|x-2|^4$, where $M$ is a bound on $|f^{(4)}[t]|$ throughout the interval determined by 2 and $x$. Therefore, $\left|\frac{-119}{24} - f[1.5]\right| = |T[1.5] - f[1.5]| \le \frac{3}{24}|1.5-2|^4 = \frac{1}{8}\cdot\frac{1}{16} = \frac{1}{128} = 0.0078125$. But $|T[1.5] - (-5)| = \left|\frac{-119}{24} + 5\right| = \frac{1}{24} > \frac{1}{128}$, so $f[1.5] = -5$ is impossible.

## ■ c.

$Q[x]$, the fourth degree Taylor polynomial about $x = 2$, for $g[x] = f[x^2+2]$ can be obtained from $T[x]$ by substituting $x^2+2$ for $x$ and truncating. We obtain $Q[x] = -3 + 5x^2 + \frac{3}{2}x^4$. The coefficient of $x$ in $Q[x]$ is $f'[0]$ and the coefficient of $x^2$ in $Q[x]$ is $\frac{g''[0]}{2}$. Hence $f'[0] = 0$ and $f''[0] = \frac{3}{2}$. By the Second Derivative Test, $f$ must have a local minimum at $x = 0$.

# Problem 5

## ■ a.

On the interval [2, 4], the graph is symmetric about the point (3, 0), so the integral over that interval is zero. Consequently, $\int_1^4 f[t]\,dt = \int_1^2 f[t]\,dt$ is the area of the trapezoid whose corners are (1, 0), (2, 0), (2, 1), and (1, 4), or $\frac{(4+1)}{2}\cdot 1 = \frac{5}{2}$. Thus, $g[4] = \frac{5}{2}$. $g[-2] = \int_1^{-2} f[t]\,dt = -\int_{-2}^{1} f[t]\,dt$ is the negative of the area of a triangle with base 3 and height 4, or $-6$.

## ■ b.

By the Fundamental Theorem of Calculus, $g'[x] = \frac{d}{dx}\int_1^x f[t]\,dt = f[x]$. Hence $g'[1] = f[1] = 4$.

## ■ c.

The absolute minimum of $g[x]$ for $-2 \le x \le 4$ lies either at a point where $g'[x] = 0$ or where $x = -2$ or where $x = 4$. We have already seen, in part a above, that $g[-2] = -6$ and that $g[4] = \frac{5}{2}$. If $g'[x] = 0$, then, by our first observation in part b above, $f[x] = 0$. This happens only at $x = -2$, which we have already computed, and at $x = 3$. But $f$, and therefore $g'$ undergoes a sign change from positive to negative as $x$ increases through 3, so $x = 3$ must give a local maximum for $g$. It follows that $g$ attains its local minimum of $-6$ at $x = -2$.

## ■ d.

If $g$ is to have an inflection point somewhere, then $g'$ must change from increasing to decreasing or from decreasing to increasing at that point. This happens when $x = 1$, but not when $x = 2$. Hence $g$ has an inflection point at just one of the two points in question.

# Problem 6

## ■ a.

An equation for the required tangent line is $y = 6 + \frac{1+e^3}{9}(x-3)$. When $x = \frac{31}{10}$, therefore, $y$ is aproximately $6 + \frac{1+e^3}{9}\cdot\frac{1}{10}$. Numerically:

```
6 + (1 + ⅇ^3)/9 0.1
```

```
6.2342837
```

## b.

One step of Euler's method is given by:

```
EulerStep[{x_, y_}, h_] := {x + h, y + (1 + ⅇ^x)/x^2 h}
```

Here is the first step from (3, 6):

```
EulerStep[{3, 6}, 0.05]
```

```
{3.05, 6.1171419}
```

And here is the second step:

```
EulerStep[%, 0.05]
```

```
{3.1, 6.2360096}
```

$f''[x]$ is

```
D[(1 + ⅇ^x)/x^2, x] // Together
```

$$\frac{-2 - 2\,e^x + e^x\,x}{x^3}$$

When $3 \leq x \leq 4$, the denominator of this fraction is positive while the numerator is $e^x(x - 2) - 2 \geq e^3 - 2 > 0$ also. Hence $f''[x] > 0$ for $3 \leq x \leq 3.1$. This means that the tangent line at each point $x \in [3.0, 3.1]$ of the curve $y = f[x]$ lies below the curve (except at the point of tangency). Euler's Method estimates the value of $y$ at $x = 3.05$ on by replacing the curve with the tangent line at (3, 6), so the estimated value of $y[3.05]$ is too small. Euler's Method then estimates $y[3.1]$ by moving along a line parallel to the actual tangent line at (3.05, $y[3.05]$), but starting at a point directly below (3.05, $y[3.05]$). The correct tangent line, if we knew it, would lead to an underestimate--again because $y''[3.05] > 0$. Because Euler's Method begins at a point below the correct point and moves parallel to the correct tangent line, the second step of Euler's Method exacerbates the existing underestimation--again in the downward direction. Hence $y[3.1] > 6.2360096$.