Solutions to the 2000 AP Calculus BC Exam Free Response Questions

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Problem 1.

a.

We must first find the point in the first quadrant where $e^{-x^2} = 1 - \cos x$. We'll call it b:

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b = x /. FindRoot[e^(-x^2) == 1 - Cos[x], {x, 1}]

0.94194408
```

The area between the two curves is $\int_0^b (e^{-x^2} - (1 - \cos x))\, dx$. We integrate numerically:

```
NIntegrate[e^(-x^2) - (1 - Cos[x]), {x, 0, b}]

0.59096245
```

b.

By the method of washers, and integrating numerically, the volume generated when R is revolved about the x-axis is $\pi \int_0^b \left[(e^{-x^2})^2 - (1 - \cos x)^2\right] dx$:

```
.π NIntegrate[(e^(-x^2))^2 - (1 - Cos[x])^2, {x, 0, b}]

1.7466141
```

■ c.

Integrating numerically, we obtain the volume of the solid described as $\int_0^b (e^{-x^2} - (1 - \cos x))^2 \, dx$.

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NIntegrate[(ⅇ^(-x^2) - (1 - Cos[x]))^2, {x, 0, b}]

0.46106351
```

Problem 2

■ a.

From the graph of runner A's velocity, which is given, we see that her velocity at time $t = 2$ is $\frac{20}{3}$ meters per second. Runner B's velocity at time t is given as $\frac{24t}{2t+3}$, so runner B's velocity at $t = 2$ is $\frac{48}{7}$ meters per second.

■ b.

Acceleration is the derivative, taken with respect to time, of velocity. In the case of runner A, at time $t = 2$, the slope of the velocity curve is $\frac{10}{3}$, so her acceleration at time $t = 2$ is $\frac{10}{3}$ meters per second per second.

```
D[24 t/(2 t + 3), t] /. t → 2

72/49
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Runner B's acceleration at $t = 2$ is $\frac{72}{49}$ meters per second per second.

■ c.

Distance travelled is the integral of velocity. Hence, reasoning from the graph of runner A's velocity, we find that runner A covered $\frac{1}{2} \cdot 3 \cdot 10 + 7 \cdot 10 = 85$ meters over the interval $0 \le t \le 10$.

$$\int_0^{10} \frac{24\,t}{2\,t + 3}\, dt$$

```
6 (20 - 3 Log[23] + Log[27])
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N[%]

83.336125
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Runner B covered 6 (20 – 3 Log[23] + Log[27]) meters, or about 83.336 meters, in the same time interval.

Problem 3.

We are given:

$$\texttt{Derivative[n\_][f\_][5]} = \frac{(-1)^n\, n!}{2^n\,(n+2)}$$

$$\frac{\left(-\frac{1}{2}\right)^n n!}{2 + n}$$

and

$$\texttt{f[5]} = \frac{1}{2}$$

$$\frac{1}{2}$$

■ a.

The third-degree Taylor polynomial for f about $x = 5$ is

```
Sum[(x - 5)^k / k! Derivative[k][f][5], {k, 0, 3}]
```

$$\frac{1}{2}+\frac{5-x}{6}+\frac{1}{16}(-5+x)^2-\frac{1}{40}(-5+x)^3$$

b.

Writing $\sum_{k=0}^{\infty} a_k (x-5)^k$ for the Taylor series in question, we have $a_k = \frac{f^{(k)}[5]}{k!} = \frac{(-1)^k}{2^k\,(k+2)}$. Thus, $\lim_{k\to\infty} \,|\, a_{k+1}(x-5)^{k+1} / a_k (x-5)^k \,| = \lim_{k\to\infty} \frac{(k+2)}{2\,(k+3)} \left| x-5 \right| = |\, x-5\,| / 2$. By the Ratio Test, the series converges absolutely when this limit is less than 1 and diverges when this limit is greater than 1. We conclude that the desired radius of convergence is 2.

c.

When $x = 6$ the Taylor series becomes $\frac{1}{2} - \frac{1}{6} + \frac{1}{16} - \frac{1}{40} + \ldots + \frac{(-1)^k}{2^k\,(k+2)} + \ldots$. As k increases, both 2^k and $k+2$ increase, so the quotients $\frac{1}{2^k\,(k+2)}$ decrease monotonically to zero as $k \to \infty$. By the Alternating Series Test, the error in replacing $f[6]$ with the sum of the first seven terms of the series is at most $\frac{1}{2^7\,(7+2)} = \frac{1}{128\cdot 9} = \frac{1}{1152} < \frac{1}{1000}$.

Problem 4

a.

The acceleration vector is the derivative of the velocity vector, taken with respect to time: $\frac{d}{dt} \langle 1 - \frac{1}{t^2}, 2 + \frac{1}{t^2} \rangle = \langle \frac{2}{t^3}, -\frac{2}{t^3} \rangle$. When $t = 3$, this is $\langle \frac{2}{27}, -\frac{2}{27} \rangle$.

b.

Position at $t = 3$ is $\langle 2, 6\rangle + \int_1^3 \langle 1 - \frac{1}{t^2}, 2 + \frac{1}{t^2} \rangle\, dt = \langle 2, 6\rangle + \langle t + \frac{1}{t}, 2\,t - \frac{1}{t} \rangle \Big|_1^3 = \langle 2, 6\rangle + \langle \frac{10}{3}, \frac{17}{3} \rangle - \langle 2, 1\rangle = \langle \frac{10}{3}, \frac{32}{3} \rangle$.

c.

Because x and y are the components of velocity, he slope of the curve at $t = t_0$ is $y[t_0] / x[t_0]$, as long as $x[t_0] \neq 0$. Thus, slope at time t_0 is $\left(2 + \frac{1}{t_0^2}\right) \Big/ \left(1 - \frac{1}{t_0^2}\right) = \frac{2\,t_0^2+1}{t_0^2-1}$. For this to be 8 we must have $2\,t_0^2 + 1 = 8\,t_0^2 - 8$, or $6\,t_0^2 = 9$. The only positive solution of this equation is $t_0 = \sqrt{3/2}$.

d.

Position at time t is $\langle 2, 6\rangle + \int_1^t \langle 1 - \frac{1}{\tau^2}, 2 + \frac{1}{\tau^2}\rangle d\tau = \langle 2, 6\rangle + \langle t + \frac{1}{t}, 2t - \frac{1}{t}\rangle - \langle 2, 1\rangle = \langle t + \frac{1}{t}, 5 + 2t - \frac{1}{t}\rangle$. As $t \to \infty$, this is asymptotic to the curve $P[t] = \langle t, 2t + 5\rangle$, which is a straight line (whose Cartesian equation is $y = 2x + 5$).

Problem 5

a.

Differentiating both sides of the equation $x y^2 - x^3 y = 6$ implicitly, we obtain $y^2 + 2 x y y' - 3 x^2 y - x^3 y' = 0$, whence $(2 x y - x^3) y' = 3 x^2 y - y^2$. Dividing both sides of the latter equation by $2 x y - x^3$, which we must assume to be non-zero, we obtain $y' = \frac{3 x^2 y - y^2}{2 x y - x^3}$, as desired.

b.

Setting $x = 1$ in the original equation, we obtain $y^2 - y = 6$, whence $y = 3$ or $y = -2$. There are therefore two points with x-coordinate 1; they are $(1, 3)$ and $(1, -2)$. At $(1, 3)$, we have $y' = \frac{3 \cdot 1^2 \cdot 3 - 3^2}{2 \cdot 1 \cdot 3 - 1^3} = \frac{0}{5} = 0$, so that an equation of the line tangent to the curve at $(1, 3)$ is $y = 3$. At $(1, -2)$, we have $y' = \frac{3 \cdot 1^2 \cdot (-2) - (-2)^2}{2 \cdot 1 \cdot (-2) - 1^3} = \frac{-10}{-5} = 2$. An equation of the line tangent to the curve at $(1, -2)$ is therefore $y = -2 + 2(x - 1)$, or $y = 2x - 4$.

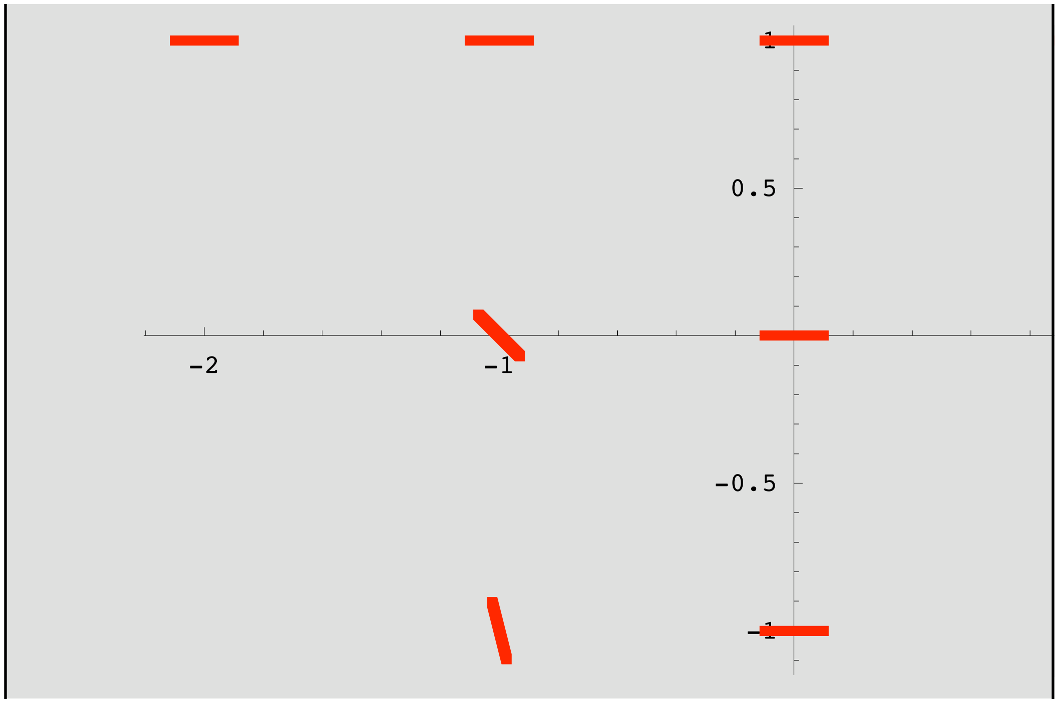
c.

We treat y as the independent variable in the original equation, and letting the prime denote differentiation with respect to y, implicit differentiation now gives us $x' y^2 + 2 x y - 3 x^2 x' y - x^3 = 0$, whence $x' = \frac{x^3 - 2 x y}{y^2 - 3 x^2 y}$. At a point with a vertical tangent, x' must vanish, so we must have $x^3 - 2 x y = 0$. Thus, $x = 0$ or $y = \frac{1}{2} x^2$. But $x y^2 - x^3 y = 6$, so $x = 0$ is not possible. On the other hand, substituting $y = \frac{1}{2} x^2$ in the original equation yields $\frac{1}{4} x^5 - \frac{1}{2} x^5 = 6$, or $x^5 = -24$. There is thus a vertical tangent at the point $\left(-\sqrt[5]{24}, \sqrt[5]{18}\right)$.

Problem 6

a.

Here is the required slope field:



b.

No solution could have the graph shown because the slope field requires any solution that passes through a point with coordinates $(x,\ 1)$ to have zero slope at that point. The graph shown has non-zero slopes at the two points where it crosses the line $y = 1$.

c.

We have $f'[x] = x\,(f[x] - 1)^2$, with $f[0] = -1$. Hence $\int_0^x \frac{f'[\xi]}{(f[\xi]-1)^2}\,d\xi = \int_0^x \xi\,d\xi$, or, by the Substitution Theorem, $\int_{-1}^y \frac{d\eta}{(\eta-1)^2} = \int_0^x \xi\,d\xi$. Thus, $-\frac{1}{2} - \frac{1}{y-1} = \frac{1}{2}\,x^2$. Solving for y, we obtain $f[x] = \frac{x^2-1}{x^2+1}$.

d.

We solve the equation $y = \frac{x^2-1}{x^2+1}$ for x^2 in terms of y and obtain $x^2 = \frac{1+y}{1-y}$. Now whatever real number x may be given, $x^2 \geq 0$, and so there is a corresponding y in the range of f precisely when $\frac{1+y}{1-y} \geq 0$. We note that $1 + y > 0$ when $y > -1$ and $1 + y < 0$ when $y < -1$, with $1 + y = 0$ when $y = -1$. On the other hand, $1 - y > 0$ when $y < 1$, $1 - y < 0$ when $y > 1$ and $1 - y = 0$ when $y = 1$. The quotient $\frac{1+y}{1-y}$ is therefore defined when $y \neq 1$; it is positive when both numerator and denominator have the same sign--that is, when $-1 < y < 1$. Consequently, the quotient is a non-negative real number when $-1 \leq y < 1$, and this is the range of f.