Solutions to the 2001 AP Calculus BC Exam

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1.

■ a.

Because the object is at (4, 5) when $t = 2$, we can obtain the slope of the line tangent to the curve when $t = 2$ by finding the value of $\frac{dy}{dx} = \left(\frac{dy}{dt}\right) \Big/ \left(\frac{dx}{dt}\right)$ at $t = 2$. We have

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m = (3 Sin[t^2]) / (Cos[t^3]) /. t → 2
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3 Sec[8] Sin[4]
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The equation of the required tangent line is thus $y = 5 + 3\,\mathrm{Sec}[8]\,\mathrm{Sin}[4]\,(x - 4)$.

■ b.

The speed of the object at $t = 2$ is the value taken on by $\sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}$ at $t = 2$. Thus, speed at $t = 2$ is

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√((3 Sin[t^2])^2 + (Cos[t^3])^2) /. t → 2
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√(Cos[8]^2 + 9 Sin[4]^2)
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■ c.

Total distance travelled over $0 \le t \le 1$ is $\int_0^1 \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$. Integrating numerically, we obtain:

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NIntegrate[(3 Sin[t^2])^2 + (Cos[t^3])^2, {t, 0, 1}]
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2.3763814
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■ d.

The position of the object at $t = 3$ is given by $\left(x[2] + \int_2^3 x'[t]\, dt,\ y[2] + \int_2^3 y'[t]\, dt\right)$. Thus, integrating numerically again, we obtain the coordinates of the position at $t = 3$:

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{4, 5} + NIntegrate[{Cos[t^3], 3 Sin[t^2]}, {t, 2, 3}]
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{3.9535019, 4.9063581}
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2.

■ a.

We have $W'[12] \simeq \frac{W[15]-W[9]}{6} = \frac{21-24}{6} = -\frac{1}{2}$ degrees Celsius/day.

■ b.

Using the Trapezoid Rule $A \simeq \frac{1}{2}(f[x_0] + 2f[x_1] + \ldots + 2f[x_{n-1}] + f[x_n])\Delta x$, the required average value is about:

$$\frac{1}{15}\,\frac{20 + 2\,31 + 2\,28 + 2\,24 + 2\,22 + 21}{2}\,3$$

$$\frac{251}{10}$$

The average water temperatureis about 25.100 degrees Celsius.

■ c.

If $P[t]$ is given by

$$P[t\_] = 20 + 10\,t\,e^{-t/3}$$

$$20 + 10\,e^{-t/3}\,t$$

then P'[t] is

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P'[t]
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$$10\,e^{-t/3} - \frac{10}{3}\,e^{-t/3}\,t$$

Putting $t = 12$, we have

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P'[12]
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$$-\frac{30}{e^4}$$

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N[%]
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-0.54946917
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On the 12th day, the water temperature is decreasing at a rate of about −0.549 degrees Celsius per day.

■ d.

The required average value is given by

$$\frac{1}{15}\int_0^{15} P[t]\,dt$$

$$\frac{1}{15}\left(90+\frac{10\,(-54+30\,e^5)}{e^5}\right)$$

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Expand[%]
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$$26-\frac{36}{e^5}$$

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N[%]
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25.757434
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Average temperature over $0 \le t \le 18$ is therefore about 25.757 degrees Celsius.

3.

■ a.

When $t = 2$, the graph shows that acceleration is 15 ft./sec. This is a positive number, so velocity is increasing when $t = 2$.

■ b.

The portion of the acceleration curve on the interval $6 \le t \le 12$ is symmetric about the point (6, 0) with the portion of the acceleration curve on the interval $0 \le t \le 6$. Consequently the integral of acceleration from 0 to 12 (which is total change in velocity over that interval) is zero. Thus, velocity is 55 ft./sec. when $t = 12$ sec.

■ c.

The car's absolute maximum velocity for $0 \le t \le 18$ is 115 ft./sec., which is the velocity it attains when $t = 6$ sec.. Velocity increases from 55 ft./sec. as long as acceleration is positive—that is, until $t = 6$. Thereafter it decreases as long as acceleration is negative—that is, while $6 \le t \le 14$. Finally, it increases again while $14 \le t \le 18$. However, the area under the acceleration curve on the latter interval is less than the area between the acceleration curve and the t-axis on the interval $6 \le t \le 14$, so the total increase in velocity that accrues while $14 \le t \le 18$ is not enough to balance out the total decrease that accrued while $6 \le t \le 14$. This means that velocity attains it absolute maximum for $0 \le t \le 18$ when $t = 6$ sec. We calculate this maximum value by finding the area of the trapezoid over the interval $0 \le t \le 6$, which is $\frac{1}{2}(2+6)\cdot 15 = 60$, and adding it to the initial velocity, 55 ft/sec to obtain a maximal velocity of 115 ft/sec.

■ d.

The car never reaches a velocity of 0 ft./sec. In fact, the absolute minimum velocity attained by the car occurs when $t = 16$, and this velocity is the initial velocity of 55 ft./sec. plus the area of the region above the t-axis over the interval [0, 6] minus the area of the region below the t-axis over the inverval [6, 16], which is $55 + 60 - 105 = 10$ ft./sec.

4.

■ a.

If $h'[x] = \frac{x^2-2}{x} = \frac{(x-\sqrt{2})(x+\sqrt{2})}{x}$, then $h'[x] = 0$ when $x = \pm\sqrt{2}$, so the graph of h has a horizontal tangent when $x = \pm\sqrt{2}$. Note that $(x+\sqrt{2})$ changes sign from negative to positive as x increases through $x = -\sqrt{2}$, while $(x-\sqrt{2})$ and x are both negative near $x = -\sqrt{2}$. Hence $h'[x]$ changes sign from negative to positive as x increases through $-\sqrt{2}$. This means that $x = -\sqrt{2}$ give a local minimum for the graph of h. On the other hand $(x-\sqrt{2})$ changes sign from negative to positive as x increases through $\sqrt{2}$, where both x and $(x+\sqrt{2})$ are positive. Hence $h'[x]$ changes sign from negative to positive as x increases through $\sqrt{2}$, and this means that h has a local minimum at $x = \sqrt{2}$ also.

■ b.

We have $h''[x] = \frac{d}{dx}(x - 2x^{-1}) = 1 + \frac{2}{x^2}$, which is always positive (except, of course, where $x = 0$). Hence h is concave upward on $(-\infty, 0)$ and on $(0, \infty)$.

■ c.

The equation of the tangent line to the graph of h at $x = 4$ is $y = h[4] + h'[4](x-4)$, or $y = (-3) + \frac{(4^2-2)}{4}(x-4)$. This is $y = \frac{7}{2}x - 17$.

■ d.

We have $h''[x] = 1 + \frac{2}{x^2}$, so that $h''[x] > 1$
for all $x > 4$.

This means that $h'[x] > h'[4] = 7/2$ for all $x > 4$.
Consequently,

$$h[x] - h[4] = \int_4^x h'[t]\,dt > \int_4^x \frac{7}{2}\,dt = \frac{7}{2}(x-4)$$

for $x > 4$. Consequently, when $x > 4$, we have $h[x] > \frac{7}{2}(x-4) + h[4] = \frac{7}{2}x - 17$, and the latter expression is the right hand side of the equation of the tangent line as found above. This means that the line tangent to the graph of $y = h[x]$ at $x = 4$ lies above the graph of h for $x > 4$.

5.

■ a.

We are given that $f'[x] = -3\,x\,f[x]$, with $f[1] = 4$ and $\lim_{x\to\infty} f[x] = 0$,
so we have

$$\int_1^\infty (-3\,x\,f[x])\,dx \;=\; \lim_{T\to\infty} \int_1^T f'[x]\,dx$$

$$=\; \lim_{T\to\infty} (f[T] - f[1])$$

$$=\; 0 \;-\; 4 \;=\; -4.$$

■ b.

We have $f'[1] == -3\cdot 1\cdot 4 = -12$, so the linearization of f at $(1, 4)$ is $L_1[x] = 4 - 12\,(x-1)$. Putting $x = \frac{3}{2}$, we obtain $L_1[\frac{3}{2}] = 4 - 6 = -2$, and we take this as the approximate value of $f[\frac{3}{2}]$. Then the approximate slope of f when $x = \frac{3}{2}$ is $f'[\frac{3}{2}] = -3\,(\frac{3}{2})\,f[\frac{3}{2}] \simeq 9$, so the linearization corresponding to $x = \frac{3}{2}$ is $L_{3/2}[x] = -2 + 9\,(x - \frac{3}{2})$. Thus, $f[2] \simeq L_{3/2}[2] = \frac{5}{2}$.

■ c.

If $f'[x] \;=\; -3\,x\,f[x]$, with $f[1] = 4$, we may write $\frac{dy}{y} = -3\,x\,dx$, whence
$\int_4^{y[x]} \frac{1}{v}\,dv = -3\int_1^x u\,du$, or $\log y[x] \;-\; \log 4 \;=\; -3\,\frac{x^2}{2} + \frac{3}{2}$. Thus, $\log y[x] = \log 4 \;+\; \frac{3}{2} - \frac{3}{2}\,x^2$, or
$y[x] = 4\,e^{(3/2)\,(1-x^2)}$.

6.

■ a.

The solution to this part will follow from later work, and will be given then.

■ b.

We have

$f[x] - \frac{1}{3} = \frac{2}{3^2}\, x + \frac{3}{3^3}\, x^2 + \ldots + \frac{n+1}{3^{n+1}}\, x^n + \ldots,$

so the series expansion for the quotient $(f[x] - \frac{1}{3})/x$ is

$\frac{2}{3^2} + \frac{3}{3^3}\, x + \ldots$. This approaches $\frac{2}{9}$ as $x \to 0$.

■ c.

Integrating term by term, we have

$\int_0^x f[t]\, dt = \frac{1}{3}\, x + \frac{1}{3^2}\, x^2 + \frac{1}{3^3}\, x^3 + \ldots + \frac{1}{3^{n+1}}\, x^{n+1} + \ldots.$

The integrated series is a geometric series with common ratio $\frac{x}{3}$, and so
$\int_0^x f[t]\, dt = \frac{x}{3-x}$, provided that $-1 < \frac{x}{3} < 1$, which is the same as $-3 < x < 3$.
We note in passing that this solves part a.): The integrated series has the same interval of convergence as the original, and so the interval of convergence for the original series is $|x| < 3$. (The problem does not ask for behavior at endpoints, and we do not consider that behavior.) The required series for this part of the problem is thus

$\int_0^1 f[t]\, dt = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots + \frac{1}{3^{n+1}} + \ldots$

■ d.

We showed in part c.) that $\int_0^x f[t]\, dt = \frac{x}{3-x}$, so $\int_0^1 f[t]\, dt = \frac{1}{3-1} = \frac{1}{2}$.