Solutions to the 2002 AP Calculus BC Exam

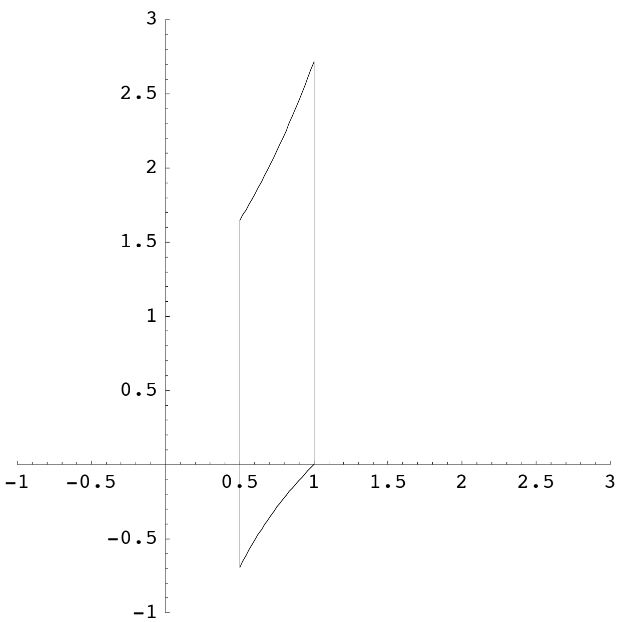
Louis A. Talman
Metropolitan State College of Denver

1.

■ a.

Here is the region between the graphs on the specified interval:

```
p1 = Plot[{ⅇ^x, Log[x]}, {x, 1/2, 1}, PlotRange → {{-1, 3}, {-1, 3}},
  AspectRatio → Automatic, DisplayFunction → Identity];
p2 = Graphics[{Line[{{1/2, ⅇ^(1/2)}, {1/2, Log[1/2]}}], Line[{{1, ⅇ}, {1, 0}}]}];
Show[p1, p2, DisplayFunction → $DisplayFunction]
```



- Graphics -

The area is given by

$$\int_{1/2}^{1} (e^x - \text{Log}[x])\, dx$$

$$\frac{1}{2}\left(1 - 2\sqrt{e} + 2e - \text{Log}[2]\right)$$

And here, for the curious, is a numeric answer:

```
N[%]
```

```
1.222987
```

■ b

Using the method of washers, we find that the required volume is

$$\pi \int_{1/2}^{1} ((4 - \text{Log}[x])^2 - (4 - e^x)^2)\, dx$$

$$\frac{1}{2}\pi\,(10 - 16\sqrt{e} + 17\,e - e^2 - 10\,\text{Log}[2] - \text{Log}[2]^2)$$

(Use integration by parts twice in succession to find $\int (\ln x)^2\, dx$.) Numerically, this is

```
N[%]
```

```
23.609493
```

■ c.

Let

$$\text{h}[\text{x\_}] = e^x - \text{Log}[x]$$

$$e^x - \text{Log}[x]$$

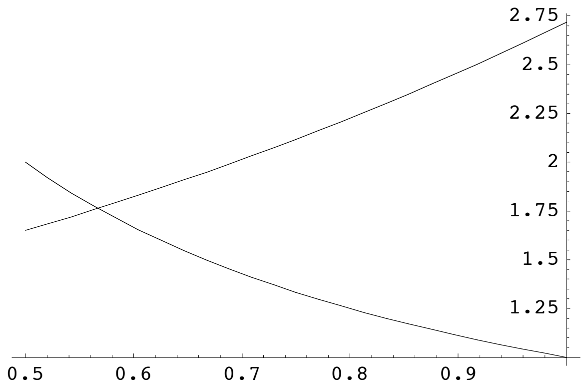
We seek the critical pointsof *h* that lie in the interval ($\frac{1}{2}$, 1) and points in the same interval where the derivative is undefined.

```
h'[x]
```

$$e^x - \frac{1}{x}$$

This is defined everywhere in ($\frac{1}{2}$, 1), so we consider only critical points. Examination of a graph

```
Plot[{e^x, 1/x}, {x, 1/2, 1}]
```




```
- Graphics -
```

indicates that there is exactly one critical point. We find it numerically:

```
b = x /. FindRoot[h'[x] == 0, {x, 0.75}]

0.56714319
```

The points of interest to us are the endpoints and the critical point, because both the absolute maximum and the absolute minimum must occur at such points. Thus we evaluate *h* at each of the points 0.5, 0.567143, and 1.0.

```
Map[h, {0.5, b, 1.0}]

{2.3418685, 2.3303661, 2.7182818}
```

The largest of these outputs is the number e itself, so it is the maximum value taken on by *h* on $[\frac{1}{2}, 1]$. The smallest output is 2.33037, so it is the minimum value taken on by *h* on $[\frac{1}{2}, 1]$.

2.

■ a.

We are given

$$\text{EnterRate[t\_]} = \frac{15600}{t^2 - 24\,t + 160}$$

$$\frac{15600}{160 - 24\,t + t^2}$$

$$\text{LeaveRate[t\_]} = \frac{9890}{t^2 - 38\,t + 370}$$

$$\frac{9890}{370 - 38\,t + t^2}$$

From these and the fact that there are no people in the park at 9:00, we have

```
NumberEntered[t_] = ∫_9^t EnterRate[τ] dτ
```

$$15600\left(\frac{1}{4}\text{ArcTan}\left[\frac{3}{4}\right]+\frac{1}{4}\text{ArcTan}\left[\frac{1}{4}(-12+t)\right]\right)$$

Consequently, at 5:00 pm,

```
NumberEntered[17.0]
```

```
6004.2703
```

so that 6004 people have entered the park by 5:00 pm.

■ b.

Revenue is given by

```
15 NumberEntered[17] + 11 (NumberEntered[23] - NumberEntered[17])
```

$$234000\left(\frac{1}{4}\text{ArcTan}\left[\frac{3}{4}\right]+\frac{1}{4}\text{ArcTan}\left[\frac{5}{4}\right]\right)+ 11\left(-15600\left(\frac{1}{4}\text{ArcTan}\left[\frac{3}{4}\right]+\frac{1}{4}\text{ArcTan}\left[\frac{5}{4}\right]\right)+15600\left(\frac{1}{4}\text{ArcTan}\left[\frac{3}{4}\right]+\frac{1}{4}\text{ArcTan}\left[\frac{11}{4}\right]\right)\right)$$

```
N[%] // InputForm
```

```
104048.16522947158
```

Rounded to the nearest dollar, as required, this gives $104,048.

■ c.

If $H[t] = \int_9^t (\text{EnterRate}[x] - \text{LeaveRate}[x])\, dx$, then $H'[t] = \text{EnterRate}[t] - \text{LeaveRate}[t]$. (This is just the Fundamental Theorem of Calculus.) Hence $H'[17]$ is given by

```
EnterRate[17] - LeaveRate[17]
```

$$-\frac{202690}{533}$$

or, numerically,

```
N[%]
```

```
-380.28143
```

$H[t]$ gives the number of people in the park at time t, where $9 \le t \le 23$. Thus, $H[17] = 3725$ is the number of people in the park at 5:00 pm. $H'[t]$ gives the rate at which the number of people in the park is increasing at time t, again for $9 \le t \le 23$. $H'[17] = -380$ means that at 5:00 pm people are leaving the park at a rate of 380 people per hour.

■ d.

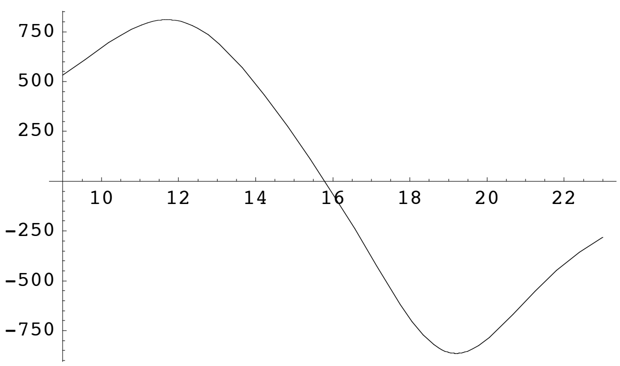
We have an expression for $H'[t]$:

```
HPrime[t_] = EnterRate[t] - LeaveRate[t] //
  Together
```

$$\frac{10\,(418960 - 35544\,t + 571\,t^2)}{(370 - 38\,t + t^2)\,(160 - 24\,t + t^2)}$$

Let's look at a picture to see if surprises are likely:

```
Plot[HPrime[t], {t, 9, 23}]
```



- Graphics -

Looks pretty simple; the maximum should occur around 4 o'clock in the afternoon. Let's nail it:

```
Solve[HPrime[t] == 0, t]
```

$$\left\{\left\{t \to \frac{4}{571}\left(4443 - \sqrt{4788614}\right)\right\}, \left\{t \to \frac{4}{571}\left(4443 + \sqrt{4788614}\right)\right\}\right\}$$

```
N[%]
```

{{t → 15.794815}, {t → 46.453872}}

The second root is outside the region of interest, but the first one is right where we thought it should be. The model therefore predicts that the number of people in the park is a maximum when $t = \frac{4}{571}\left(4443 - \sqrt{4788614}\right)$.

3.

■ a.

The slope m of a curve given parametrically by $x = x[t]$, $y = y[t]$ at $t = t_0$ is given by $m = y'[t_0]/x'[t_0]$. Hence, the slope of the roller-coaster track at $t = 2$ is $m[2] = y'[2]/x'[2] = (18 \sin[2] + \cos[2] - 1)/(10 + 4\cos[2])$. For those who like numeric answers, this is

```
18 Sin[2] + Cos[2] - 1
────────────────────── // N
    10 + 4 Cos[2]
```

```
1.7936973
```

■ b.

The acceleration vector along a curve $x = x[t]$, $y = y[t]$ is $\{x''[t], y''[t]\}$. Thus, the required acceleration vector is given by the following:

```
x[t_] = 10 t + 4 Sin[t]
```

```
10 t + 4 Sin[t]
```

```
y[t_] = (20 - t) (1 - Cos[t])
```

```
(20 - t) (1 - Cos[t])
```

We need to know the value of t that gives $x[t] = 140$.

```
FindRoot[x[t] == 140, {t, 15}]
```

```
{t → 13.647083}
```

Let's call it T.

```
T = t /. %
```

```
13.647083
```

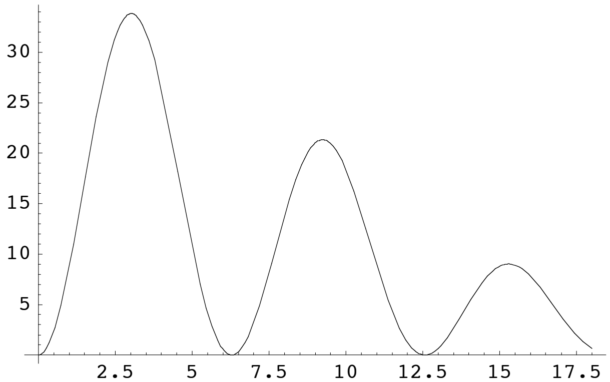
```
{x''[T], y''[T]}
```

```
{-3.5291729, 1.225733}
```

■ c.

We seek first the maximum height--that is, the maximum value for $y[t]$ over the interval $0 \le t \le 18$. To this end, we find the critical points for $y[t]$. (There are no points where $y'[t]$ is undefined in the specified interval.) Let us first look at a graph of $y[t]$ to see about where we should look:

```
Plot[y[t], {t, 0, 18}]
```




```
- Graphics -
```

Maximal height appears to occur somewhere around $t = 3$. Let's nail it down:

```
FindRoot[y'[t] == 0, {t, 3}]
```

```
{t → 3.0239158}
```

And let's give that number a name:

t_{max} = t /. %

```
3.0239158
```

Maximal height occurs at about 3.024 seconds.

Speed is $|v[t]| = |\{x'[t], y'[t]\}| = \sqrt{x'[t]^2 + y'[t]^2}$, so now we want

$\sqrt{x'[t_{max}]^2 + y'[t_{max}]^2}$

```
6.0276637
```

■ d.

Because $y[t] = (20 - t)(1 - \cos t)$, we see that $y = 0$ for $0 < t < 18$ only when $\cos t = 1$. This happens only at $t_1 = 2\pi$ and at $t_2 = 4\pi$. The average speed for $t_1 \le t \le t_2$ is $\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} s[\tau]\, d\tau = \frac{1}{2\pi} \int_{2\pi}^{4\pi} \sqrt{x'[\tau]^2 + y'[\tau]^2}\; d\tau = \frac{1}{2\pi} \int_{2\pi}^{4\pi} \sqrt{(10 + 4\cos\tau)^2 + ((20 - \tau)\sin\tau + \cos\tau - 1)^2}\; d\tau.$

Note: Numeric integration (not required) gives an average speed of

$\frac{1}{2\pi}$ NIntegrate$\left[\sqrt{x'[t]^2 + y'[t]^2}, \{t, 2\pi, 4\pi\}\right]$

```
12.504401
```

4.

a.

The function f, given pictorially, can be written as $f[x] = -3 + 3(x+2) = 3x + 3$ for $-2 \le x \le 0$; $f[x] = -3x+3$ for $0 \le x \le 2$. Then $g[x] = \int_0^x (3t+3)\,dt = 3x + \frac{3}{2}x^2$ when $-2 \le x \le 0$, and $g[x] = \int_0^x (-3t+3)\,dt = 3x - \frac{3}{2}x^2$ when $0 < x < 2$. Thus, $g[-1] = -3 + \frac{3}{2} = \frac{-3}{2}$. (We could also have found this by looking at the area of the appropriate triangle in the figure.) By the Fundamental Theorem of Calculus, $g'[-1] = f[-1] = 0$. Also by the Fundamental Theorem of Calculus, $g''[-1] = f'[-1] = 3$.

b.

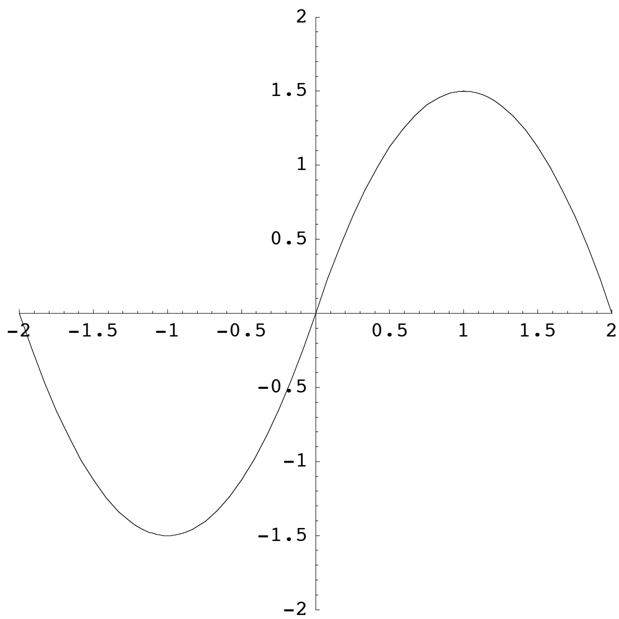
As we noted above for the special case $t = -1$, the Fundamental Theorem of Calculus tells us that $g'[t] = f[t]$. Thus, g is increasing on the closures of those intervals where $f[t] > 0$. From the picture, we conclude that g is an increasing function on the interval $[-1, 1]$, and on that interval only.

c.

From $g'[t] = f[t]$, which we derived above, we conclude that $g''[t] = f'[t]$. Now $f'[t] = 3$ for $-2 < t < 0$, and $f'[t] = -3$ for $0 < t < 2$. We conclude that g is concave downward on the interval $(0, 2)$, where $f'[t] < 0$.

■ d.

```
p1 = Plot[3 x + 3/2 x^2, {x, -2, 0}, DisplayFunction → Identity];
p2 = Plot[3 x - 3/2 x^2, {x, 0, 2}, DisplayFunction → Identity];
Show[p1, p2, PlotRange → {{-2, 2}, {-2, 2}},
 AspectRatio → Automatic, DisplayFunction → $DisplayFunction]
```

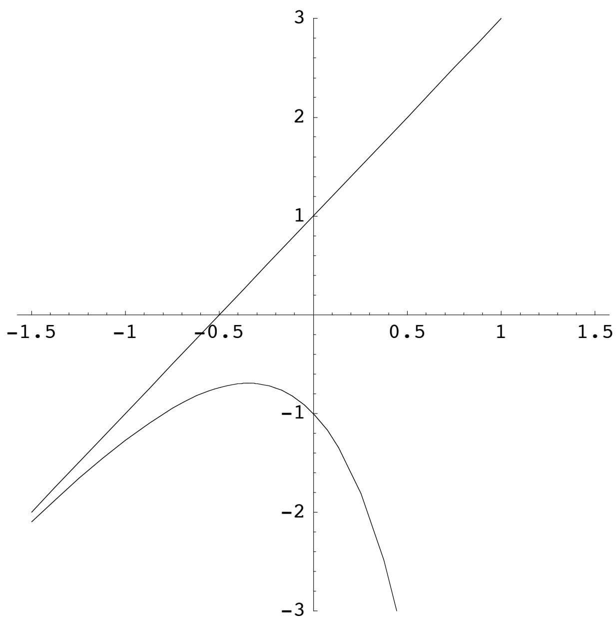


- Graphics -

5.

■ a.

The curves should look like this:



■ b.

If $f[0] = 1$, then $f'[0] = 2 \cdot 1 - 4 \cdot 0 = 2$, so $f[0.1] \sim f[0] + f'[0] \cdot 0.1 = 1 + 2 \cdot 0.1 = 1.2$. Thus, $f'[0.1] \sim 2 \cdot f[0.1] - 4 \cdot 0.1 \sim 2 \cdot 1.2 - 4 \cdot 0.1 = 2$. Hence, $f[0.2] \sim f[0.1] + f'[0.1] \cdot 0.1 \sim 1.2 + 2 \cdot 0.1 = 1.4$. The Euler approximation to the value $f[0.2]$ of the solution at $x = 0.2$, using step-size 0.1, is 1.4.

■ c.

If $y = 2x + b$ is to be a solution, then we must have $2 = y' = 2y - 4x = 2(2x + b) - 4x = 2b$. Consequently, $b = 1$.

■ d.

If g is the function that satisfies the equation $y' = 2y - 4x$ with initial condition $g[0] = 0$, then $g'[0] = 2 \cdot 0 - 4 \cdot 0 = 0$, so g has a critical point at $x = 0$. Furthermore, $g''[x] = \frac{d}{dx}(2y - 4x) = 2g'[x] - 4$, so that $g''[0] = 2g'[0] - 4 = -4$. Consequently, by the Second Derivative Test, g has a local maximum at (0, 0).

6.

■ a & b.

If $f[x] = \sum_{n=0}^{\infty} \frac{(2x)^{n+1}}{n+1}$ inside the interval of convergence of the series, then term-by-term differentiation gives $f'[x] = \sum_{n=0}^{\infty} 2(2x)^n$ in the same region. But this is a geometric series, whose interval of convergence is therefore $(\frac{-1}{2}, \frac{1}{2})$. (That is, where $|2x| < 1$.) We thus have $f'[x] = 2 + 4x + 8x^2 + 16x^3 + \ldots + 2(2x)^n + \ldots.$ (This is part b.) In order to finish part a, we must decide about convergence of the original series when $x = \pm\frac{1}{2}$. (We already know that the radius of convergence is $\frac{1}{2}$ because the radius of convergence of the derived series is always the same as the radius of convergence of the original series.) But when $x = \frac{1}{2}$, the original series becomes the harmonic series, which diverges, and when $x = -\frac{1}{2}$, the series becomes the alternating harmonic series, which diverges. The interval of convergence is thus $[-\frac{1}{2}, \frac{1}{2})$.

■ c.

We have seen above that $f'[x] = 2\sum_{n=0}^{\infty}(2x)^n$ is a geometric series. Consequently, when $-\frac{1}{2} < x < \frac{1}{2}$, we have $f[x] = \frac{2}{1-2x}$. It follows that $f[-\frac{1}{3}] = \frac{2}{1+\frac{2}{3}} = \frac{6}{5}$.