

Solutions to the 2002 AP Calculus BC Exam (Form B)

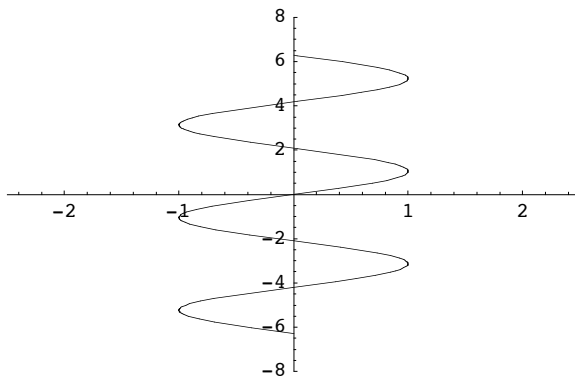
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1.

■ a.

The graph should look like this:

```
In[1]:= x[t_] = Sin[3 t]; y[t_] = 2 t;
ParametricPlot[{x[t], y[t]}, {t, -π, π}, PlotRange → {{-2.5, 2.5}, {-8, 8}}]
```



Out[1]= - Graphics -

■ b

The range of $x[t] = \sin 3t$ is $-1 \leq x \leq 1$. The range of $y[t] = 2t$ is $-2\pi \leq y \leq 2\pi$.

■ c.

$x[t] = \sin 3t$ reaches its positive maximum of 1 at $t = -\frac{\pi}{2}$, at $t = \frac{\pi}{6}$, and at $t = \frac{5\pi}{6}$ in the interval $-\pi \leq t \leq \pi$. Thus, the smallest positive value of t for which $x[t]$ is maximal here is $t = \frac{\pi}{6}$. Speed is the length of the velocity vector $v[t] = \{x'[t], y'[t]\}$, or

$$\text{In[2]} := \sqrt{\mathbf{x}'[t]^2 + \mathbf{y}'[t]^2}$$

$$\text{Out[2]} = \sqrt{4 + 9 \text{Cos}[3 t]^2}$$

```
In[3]:= % /. t ->  $\frac{\pi}{6}$ 
```

```
Out[3]= 2
```

The speed of the particle at $t = \frac{\pi}{6}$ is 2.

■ d.

Speed, found above is $\sqrt{4 + 9 \cos^2 3t}$. Consequently, distance travelled during the time interval $-\pi \leq t \leq \pi$ is $\int_{-\pi}^{\pi} \sqrt{4 + 9 \cos^2 3t} dt$. Here is a numeric approximation:

```
In[4]:= NIntegrate[ $\sqrt{x'[t]^2 + y'[t]^2}$ , {t, - $\pi$ ,  $\pi$ }]
```

```
Out[4]= 17.9734266849
```

And here is a numeric approximation of 5π :

```
In[5]:= N[5  $\pi$ ]
```

```
Out[5]= 15.7079632679
```

The particle clearly travels a distance that is greater than 5π .

2.

■ a.

We are given

```
In[6]:= P'[t_] = 1 - 3 e-0.2  $\sqrt{t}$ 
```

```
Out[6]= 1 - 3 e-0.2  $\sqrt{t}$ 
```

```
In[7]:= P'[9]
```

```
Out[7]= -0.646434908282
```

Now $P'[9] < 0$, so the amount of pollutant in the lake is decreasing when $t = 9$.

■ b.

Because the exponential $e^{-0.2\sqrt{t}}$ decreases from 1 to 0 on $0 \leq t < \infty$, $P'[t]$ is an increasing function on that interval. Moreover, $P'[0] = -2$ and $\lim_{t \rightarrow \infty} P'[t] = 1$. Because P' is a continuous, strictly increasing function on $0 \leq t < \infty$, it follows P' has exactly one zero, and that P therefore has exactly one critical point, in the interval--where, moreover, P' changes sign from negative to positive. Thus, this zero of P' gives an absolute minimum for P . We solve to find that zero:

```
In[8]:= FindRoot[P'[t] == 0, {t, 20}]
```

```
Out[8]= {t -> 30.1737240203}
```

The amount of pollutant in the lake is at a minimum when $t \sim 30.174$ days. Let's call that number T .

```
In[9]:= T = t /. %
```

```
Out[9]= 30.1737240203
```

■ c.

By the Fundamental Theorem of Calculus, the amount $P[t]$ of pollutant in the lake satisfies $P[t] - P[0] = \int_0^t P'[\tau] d\tau$, or $P[t] = 50 + \int_0^t P'[\tau] d\tau$.

Integrating numerically, we find that $P[T]$ is about

```
In[10]:= 50 + NIntegrate[P'[t], {t, 0, T}]
```

```
Out[10]= 35.1043384535
```

Because the lake is considered safe when $P \leq 40$, and $P[T] \sim 35.104$, we conclude that the lake is safe when the amount of pollutant in it reaches its minimum.

■ d.

The linearization L at $t = 0$ has equation

```
In[11]:= L[t_] = 50 + P'[0] t
```

```
Out[11]= 50 - 2 t
```

Thus, the linearization predicts the arrival of safety when $50 - 2t = 40$, or when $t = 5$.

3.

■ a.

We begin by finding the x -coordinate of the intersection point:

```
In[12]:= NSolve[3/4 x == 4 x - x^3 + 1, x]
```

```
Out[12]= {{x -> -1.62290512164}, {x -> -0.317544438557}, {x -> 1.9404495602}}
```

We want the positive root. We'll call it b :

```
In[13]:= b = x /. % [3]
```

```
Out[13]= 1.9404495602
```

The area we need is now $\int_0^b ((4x - x^3 + 1) - \frac{3}{4}x) dx$:

```
In[14]:= Integrate[(4 x - x^3 + 1) - (3/4 x), x, {x, 0, b}]
```

```
Out[14]= 4.51467957289
```

■ b.

We can use the method of washers to find the required volume:

```
In[15]:= Pi Integrate[(4 x - x^3 + 1)^2 - (3/4 x)^2, x, {x, 0, b}]
```

```
Out[15]= 57.4632682799
```

■ c.

The perimeter of the region is $1 + \frac{5}{4}b + \int_0^b \sqrt{1 + (4 - 3x^2)^2} dx$.

4.

■ a.

If $g[x] = 5 + \int_6^x f[t] dt$, then $g[6] = 5$, because $\int_6^6 f[t] dt = 0$. By the Fundamental Theorem of Calculus, $g'[x] = f[x]$, so $g'[6] = 3$. (We have read the value of $f[6]$ from the graph given in the statement of the problem.) We now see that $g''[x] = f'[x]$, and we have been given that $f'[6] = 0$ (in the equivalent form that the line tangent to the curve at $x = 6$ is horizontal). Hence $g''[6] = 0$.

■ b.

We know that $g'[x] = f[x]$, and we also know that g is decreasing on any interval where g' is negative. Moreover, if a continuous function is decreasing on an open interval, it is also decreasing on the closure of that interval. From the graph given, we see that $f[x] < 0$ on $[-3, 0)$ and on $(12, 15]$. We conclude that g is decreasing on $[-3, 0]$ and that g is decreasing on $[12, 15]$.

Note: We may not conclude that g is decreasing on $[-3, 0] \cup [12, 15]$. In fact, values of $g[x]$ for x in $[-3, 0]$ will be negative and smaller than those for x in $[12, 15]$ --which will be positive.

■ c.

A function is concave downward on any open interval where its derivative is decreasing. But $g'[x] = f[x]$ is decreasing on the interval $[6, 15]$, and nowhere else. Thus, g is concave downward on $([6, 15])$. Whether or not to include the endpoints of this interval is highly dependant upon the definition one uses for "concave downward". There are several distinct definitions in use in different textbooks on the market, and the end-points should not affect scoring.

■ d.

The required trapezoidal approximation for $\int_{-3}^{15} f[t] dt$ is $(f[-3] + 2f[0] + 2f[3] + 2f[6] + 2f[9] + 2f[12] + f[15]) \cdot \frac{3}{2}$.

Reading function values from the graph, we obtain $(-1 + 0 + 2 + 6 + 2 + 0 + -1) \cdot \frac{3}{2} = 12$.

5.

■ a.

If the line $y = -2$ is tangent to the solution curve $y = f[x]$ of the differential equation $y' = \frac{3-x}{y}$, then at the point of tangency we must have $y' = 0$, or $\frac{3-x}{y} = 0$. This means that the x -coordinate of the point of tangency must be 3. Because the y -coordinate of the point of tangency is necessarily -2 , there is then an open interval I centered at $x = 3$ throughout which $f[x] < 0$. We may assume that I does not contain 0. (This is a consequence of the continuity of f , which we infer from its differentiability.) If $x \in I$ and $x < 3$, then $y' = \frac{3-x}{y} < 0$ because $3 - x > 0$ and $y < 0$. If, on the other hand, $x \in I$ and $x > 3$, then $y' = \frac{3-x}{y} > 0$, because $3 - x < 0$ and $y < 0$. Consequently, $x = 3$ is a critical point for f and the derivative of f is negative to the right but positive to the left of this critical point. It follows, by the First Derivative Test, that f has a local minimum at $x = 3$.

Note: This problem can also be solved by finding the solution of the differential equation that passes through the point $(3, -2)$. This solution, which can be found by separation of variables, as below, is $f[x] = -\sqrt{-(x^2 - 6x + 5)}$. Then $f'[x] = \frac{x-3}{\sqrt{-(x^2-6x+5)}}$. Either the First Derivative Test or the Second Derivative Test applies here; the First Derivative Test is more efficient.

■ b.

If $f'[x] = (3 - x)/f[x]$, then $f[x]f'[x] = 3 - x$. Consequently, $\int_6^x f[\tau]f'[\tau] d\tau = \int_6^x (3 - \tau) d\tau$.

Hence $\frac{1}{2}f[x]^2 - \frac{1}{2}f[6]^2 = (3x - \frac{1}{2}x^2) - (18 - 18)$, or $f[x]^2 = 16 + 6x - x^2$. Consequently, $f[x] = -\sqrt{16 + 6x - x^2}$, where we have chosen the minus sign for the square root to reflect that $f[6] = -4$.

6.**■ a.**

We are given that $\sum_{k=1}^{\infty} \frac{u^k}{k} = \ln\left(\frac{1}{1-u}\right)$. Hence, $\ln\left(\frac{1}{1+3x}\right) = \ln\left(\frac{1}{1-(-3x)}\right) = \sum_{k=1}^{\infty} \frac{(-3x)^k}{k} = \sum_{k=1}^{\infty} (-1)^k \frac{3^k}{k} x^k$. Because the interval of convergence for the first series is $-1 \leq u < 1$, the second series converges for $-1 \leq -3x < 1$, or for $\frac{1}{3} < x \leq -\frac{1}{3}$.

■ b.

We have $\sum_{k=1}^{\infty} \frac{(-1)^k}{k} = \ln\left(\frac{1}{1-(-1)}\right) = \ln\left(\frac{1}{2}\right) = -\ln 2$.

■ c.

We know that $\sum_{k=1}^{\infty} \frac{1}{k^q}$ converges if $q > 1$, but diverges if $q \leq 1$. Consequently, if we want $\sum_{k=1}^{\infty} \frac{1}{k^{2p}}$ to diverge, we will need to have $2p \leq 1$. On the other hand, $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^p}$ is an alternating series, and it will converge if the magnitudes of the terms decrease to zero. This will be so if $p > 0$. Thus, $p = \frac{1}{2}$ will work--as will any p such that $0 < p \leq \frac{1}{2}$.

■ d.

Here, reasoning as above, we need to have $p \leq 1$ in order to have $\sum_{k=1}^{\infty} \frac{1}{k^p}$ diverge while we need to have $2p > 1$ in order to have $\sum_{k=1}^{\infty} \frac{1}{k^{2p}}$ converge. Now $p = 1$ will work--as will any p such that $\frac{1}{2} < p \leq 1$.