# Solutions to the 2003 AP Calculus BC Exam

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## Problem 1.

### ■ a.

The curves intersect where $\sqrt{x} = e^{-3x}$:

$$a = x \,/.\, \text{FindRoot}\left[\sqrt{x} == e^{-3x},\ \left\{x,\ \frac{1}{2}\right\}\right][[1]]$$

```
0.2387341
```

The area of the region $R$ is thus

$$\int_a^1 \left(\sqrt{x} - e^{-3x}\right) dx$$

```
0.44262992
```

### ■ b.

This problem is most easily solved using the method of washers. The required volume is then

$$\pi \int_a^1 \left((1 - e^{-3x})^2 - \left(1 - \sqrt{x}\right)^2\right) dx$$

```
1.4235585
```

It is also possible to use the method of shells. This method leads to

$$2\pi \int_{e^{-3}}^{\sqrt{a}} (1 - y) \left(1 + \frac{1}{3} \text{Log}[y]\right) dy + 2\pi \int_{\sqrt{a}}^1 (1 - y)\,(1 - y^2)\, dy$$

```
1.4235585
```

### ■ c.

The area of the cross section meeting the $x$-axis at $x = h$ is

```
A[h_] = 5 (√h - e^(-3 h))^2 // Expand
```

$5\,e^{-6\,h} - 10\,e^{-3\,h}\,\sqrt{h} + 5\,h$

When we attempt to find $\int_a^1 A[x]\,dx$, we find that the middle term requires numeric integration. Consequently, the required volume is

```
NIntegrate[A[x], {x, a, 1}]
```

1.5543544

## Problem 2

### ■ a.

The quantity $x'[t] = -9\cos(\frac{\pi t}{6})\sin\left(\frac{\pi\sqrt{t+1}}{2}\right)$ is negative on both of the intervals $(0,\ 3)$ and $(3,\ 9]$. This is because $\frac{\pi t}{6}$ lies between 0 and $\frac{\pi}{2}$ when $0 < t < 3$, but between $\frac{\pi}{2}$ and $\pi$ when $3 < t < 9$, making the cosine factor in $x'[t]$ positive when $0 < t < 3$ and negative when $3 < t < 9$. On the other hand, $\frac{\pi\sqrt{t+1}}{2}$ lies between 0 and $\pi$ when $0 < t < 3$, but between $\pi$ and $2\pi$ when $3 < t < 9$, making the sine factor positive for $0 < t < 3$ and negative for $3 < t < 9$. Note that both factors are 0 when $t = 3$, and this must be the value of $t$ which produces that cusp that appears in the picture at the point $B$. Hence $3 < t < 9$ when the particle is at $C$. This means that $x'[t] < 0$ when the particle is at $C$. The slope of the tangent line at $C$ to the curve shown is positive, and that slope is given by $\frac{y'[t]}{x'[t]}$. Because $x'[t]$ and $y'[t]$ have the same sign, it follows that $y'[t] < 0$when the particle is at $C$.

### ■ b.

We indicated in part a. above that the particle reaches the cusp at $B$ when $t = 3$. This follows from the fact that the slope of the tangent line at a point $(x[t],\ y[t])$ is $\frac{y'[t]}{x'[t]}$, and must be undefined at $B$. Because $y[t]$ is given differentiable, this requires that $x'[t] = 0$, which occurs only at $t = 3$.

### ■ c.

The slope of the tangent line at $(x[8],\ y[8])$ is $\frac{5}{9}$, so $y'[8] = \frac{5}{9}\,x'[8]$. Now $x'[8]$ is

```
-9 Cos[π t / 6] Sin[π √(t + 1) / 2] /. t → 8
```

$-\frac{9}{2}$

Hence $y'[8] = \frac{5}{9}\,(-\frac{9}{2})$, and the velocity vector is $\langle -\frac{9}{2},\ -\frac{5}{2}\rangle$. Speed when $t = 8$ is therefore $\sqrt{\frac{53}{2}}$.

### ■ d.

Distance from $A$ to $D$ is $|x[9] - x[0]|$, or $\left|\int_0^9 x'[t]\,dt\right|$. Integrating numerically, we obtain

```
Abs[-9 NIntegrate[Cos[π t / 6] Sin[π Sqrt[t + 1] / 2], {t, 0, 9}]]
```

```
39.255369
```

## Problem 3.

### ■ a.

At the point $P$ we must have $\frac{5}{3}\,y = \sqrt{1+y^2}$, which implies that $25\,y^2 = 9 + 9\,y^2$ or $y^2 = \frac{9}{16}$. Rejecting the extraneous negative solution for $y$, we obtain $y = \frac{3}{4}$. Thus, at $P$ we have $y = \frac{3}{4}$, and $x = \frac{5}{3}\,y = \frac{5}{4}$. On the curve $C$, we have

```
x[y_] = Sqrt[1 + y^2]
```

$\sqrt{1 + y^2}$

```
x'[y]
```

$\frac{y}{\sqrt{1 + y^2}}$

```
x'[5/4]
```

$\frac{5}{\sqrt{41}}$

### ■ b.

$$\int_0^{3/4} \left(\sqrt{1 + y^2} - \frac{5}{3}\,y\right) dy$$

$\frac{1}{2}\,\text{ArcSinh}\left[\frac{3}{4}\right]$

(Note: The first integral can be obtained by making the substitution $y = \sinh u$; $dy = \cosh u\,du$.)

### ■ c.

The relations $x = r\cos\theta$, $y = r\sin\theta$ give the transformation from rectangular to polar coordinates. Subtituting these relations for $x$ and $y$ in the equation $x^2 - y^2 = 1$ yields $r^2\cos^2\theta - r^2\sin^2\theta = 1$, which is equivalent to $r^2 = \frac{1}{\cos^2\theta - \sin^2\theta}$.

### ■ d.

(Note: The copy of the exam distributed on-lin by ETS appears to be incomplete here. A phrase is missing from this question, and I assume that we are asked for an integral in polar coordinates *that gives the area of the region S*.)

The line $x = \frac{5}{3}\,y$ can be written as $r\cos\theta = \frac{5}{3}\,r\sin\theta$, which reduces to $\tan\theta = \frac{3}{5}$, or, equivalently, $\theta = \text{Arctan}\,\frac{3}{5}$. Thus, the required integral is $\frac{1}{2}\int_0^{\text{Arctan}[3/5]} r^2\,d\theta = \frac{1}{2}\int_0^{\text{Arctan}[3/5]} \frac{1}{\cos^2\theta - \sin^2\theta}\,d\theta = \frac{1}{2}\int_0^{\text{Arctan}[3/5]} \sec 2\,\theta\,d\theta$. Note that this integral is easier to evaluate than that which appears in part b, above.

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## Problem 4

### ■ a.

The graph $f'[x]$, as given, lies above the $x$-axis only on the interval $[-3, -2)$, so $f$ is increasing on $[-3, -2]$.

### ■ b.

Inflection points are places where the derivative changes from increasing to decreasing or from decreasing to increasing. For this function, we see from the graph of $f'$ that this occurs when $x = 0$ and when $x = 2$.

### ■ c.

We have $f'[0] = -2$, so the tangent line at (0, 3) has equation $y = 3 - 2\,x$.

### ■ d.

We know by the Fundamental Theorem of Calculus that $f[x] = 3 + \int_0^x f'[t]\,dt$, so $f[-3] = 3 + \int_0^{-3} f'[t]\,dt$. Now $\int_{-3}^0 f[t]\,dt$ is the area of a triangle of base 1 and height 1 minus the area of a triangle of base 2 and height 2, or $\frac{1}{2} - 2 = -\frac{3}{2}$. Thus $f[-3] = 3 + \frac{3}{2} = \frac{9}{2}$. On the other hand, $f[4] = 3 + \int_0^4 f[t]\,dt$, and $\int_0^4 f[t]\,dt$ is the negative of the area that remains when a semi-circle of radius 2 is removed from a rectangle of base 4 and height 2; this is $-(8 - 2\,\pi)$. Thus $f[4] = 3 - (8 - 2\,\pi) = 2\,\pi - 5$.

# Problem 5

## ■ a.

Because $V = \pi r^2 h = 25 \pi h$, $\frac{dV}{dt} = 25 \pi \frac{dh}{dt}$. But we are given that $\frac{dV}{dt} = -5 \pi \sqrt{h}$. Hence $25 \pi \frac{dh}{dt} = -5 \pi \sqrt{h}$, and, dividing by $25 \pi$, we obtain $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$.

## ■ b.

We have $\frac{1}{\sqrt{h}} \frac{dh}{dt} = -\frac{1}{5}$, so $\int_0^t \frac{1}{\sqrt{h[\tau]}} h'[\tau] \, d\tau = -\int_0^t \frac{d\tau}{5}$. Substituting $u = h[\tau]$ in the integral on the left, we obtain $\int_{17}^{h[t]} \frac{du}{\sqrt{u}} = -\int_0^t \frac{d\tau}{5}$, or $2 \sqrt{h[t]} - 2 \sqrt{17} = -\frac{1}{5} t$. Thus $h[t] = \left(\sqrt{17} - \frac{1}{10} t\right)^2$.

## ■ c.

The coffee pot is empty when $h[t] = \left(\sqrt{17} - \frac{1}{10} t\right)^2 = 0$, or when $t = 10 \sqrt{17}$ seconds.

# Problem 6

## ■ a.

When $f[x] = \sum_{k=0}^{\infty} a_k x^k$, we have $a_k = \frac{f^{(k)}[0]}{k!}$. Thus $f'[0] = a_1 = 0$, and $f''[0] = 2 a_2 = -\frac{1}{3}$. So $f[x]$ has a critical point at $x = 0$ and $f''[0] < 0$. By the Second Derivative Test, $f$ has a local maximum at $x = 0$.

## ■ b.

When $x = 1$, we have $f[1] = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \ldots$. The denominators increase as we move to the right in the series, and this is an alternating series. Hence, by the Alternating Series Test, the error in approximating $f[1]$ by $1 - \frac{1}{3!}$ is no larger than $\frac{1}{5!} = \frac{1}{120} < \frac{1}{100}$.

## ■ c.

Note that $x f[x] = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sin x$. Hence $f[x] = \frac{\sin x}{x}$, extended through the origin by continuity. Thus, $f'[x] = \frac{x \cos x - \sin x}{x^2}$, also extended through the origin by continuity, and $x f'[x] + y = x \frac{x \cos x - \sin x}{x^2} + \frac{\sin x}{x} = \frac{x \cos x - \sin x + \sin x}{x} = \cos x$.