

Solutions to the 2003 AP Calculus BC Exam (Form B)

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Problem 1.

■ a.

If $f[x] = 4x^2 - x^3$, and $g[x] = 18 - 3x$, then the curves have an intersection in the first quadrant where $x = 3$.

$$\text{Solve}[4x^2 - x^3 == 18 - 3x, x]$$

$$\{\{x \rightarrow -2\}, \{x \rightarrow 3\}, \{x \rightarrow 3\}\}$$

We have $f'[x] = 8x - 3x^2$, so the equation of the line tangent to the curve $y = f[x]$ at $x = 3$ is
 $y = f[3] + f'[3](x - 3)$,
 or $y = (4 \cdot 3^2 - 3^3) + (8 \cdot 3 - 3 \cdot 3^2)(x - 3) = 9 + (24 - 27)(x - 3)$. This is $y = 9 - 3x + 9$, or $y = 18 - 3x$.

■ b.

The curve $y = f[x]$ intersects the x -axis at $x = 4$, and the line crosses the x -axis at $x = 6$. Therefore the area of the region S is

$$\int_3^4 ((18 - 3x) - (4x^2 - x^3)) dx + \int_4^6 (18 - 3x) dx$$

$$\frac{95}{12}$$

■ c.

The curve $y = 4x^2 - x^3$ crosses the x -axis at $x = 0$ and at $x = 4$. Thus, the volume generated when the region R is revolved about the x -axis is

$$\pi \int_0^4 (4x^2 - x^3)^2 dx$$

$$\frac{16384 \pi}{105}$$

(Note: I mistakenly found the area generated by the region S instead of the region R in an earlier version of these solutions. Thanks to Heather Kelly for alerting me to the mistake.)

Problem 2

■ a.

The circle of radius 2 has equation $y = \sqrt{2 - x^2}$ in the first quadrant and extends over the interval $0 \leq x \leq \sqrt{2}$ there. The circle of radius 1 has equation $y = \sqrt{1 - (x - 1)^2}$ in the first quadrant and extends over the interval $0 \leq x \leq 2$ there. Using the fact that the two semi-circles intersect at $x = 1$, the required expression is therefore

$$\int_0^1 \sqrt{1 - (x - 1)^2} dx + \int_1^{\sqrt{2}} \sqrt{2 - x^2} dx.$$

■ b.

The circle of radius 2 has equation $x = \sqrt{2 - y^2}$ in the first quadrant and extends over the interval $0 \leq y \leq \sqrt{2}$ there. The first-quadrant portion of the circle of radius 1 that lies inside the larger circle has equation $x = 1 - \sqrt{1 - y^2}$ and extends over the interval $0 \leq y \leq 1$. The required expression is therefore $\int_0^1 (\sqrt{2 - y^2} - 1 + \sqrt{1 - y^2}) dy$.

■ c.

Note: A phrase seems to have been left out of the distributed version of this question. I assume that the blank space between the symbol " θ " and the symbol " R " should contain the phrase "that represents the area of", or some phrase of similar meaning.

The segment connecting the origin to the first-quadrant intersection point of the two circles, given as $(1, 1)$ lies in the ray $\theta = \frac{\pi}{4}$. The area of the sector of the large circle that lies below this segment but above the positive x -axis is a part of R and has area $\frac{1}{2} \int_0^{\pi/4} r^2 d\theta = \int_0^{\pi/4} d\theta$. The remainder of the region R after this sector is removed has bounding curve $r = 2 \cos \theta$, where $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$. Hence, its area is $\frac{1}{2} \int_{\pi/4}^{\pi/2} r^2 d\theta = 2 \int_{\pi/4}^{\pi/2} \cos^2 \theta d\theta$. The required expression is therefore $\int_0^{\pi/4} d\theta + 2 \int_{\pi/4}^{\pi/2} \cos^2 \theta d\theta$.

Problem 3.

■ a.

Average radius is $\frac{1}{720} \int_0^{360} B[x] dx$.

■ b.

The required midpoint Riemann sum is

$$\frac{1}{720} (30 (120 - 0) + 30 (240 - 120) + 24 (360 - 240))$$

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■ **c.**

The integral $\pi \int_{125}^{275} \left(\frac{B[x]}{2}\right)^2 dx$ gives the volume, in cubic millimeters, of the segment of the blood vessel that extends from $x = 125\text{mm}$ to $x = 275\text{ mm}$.

■ **d.**

We have $B[60] = B[180]$. Consequently, by Rolle's Theorem, there is a number $x_1 \in (60, 180)$ where $B'[x_1] = 0$. We also have $B[240] = B[360]$, so, once again by Rolle's Theorem, there is a number $x_2 \in (240, 360)$ where $B'[x_2] = 0$. But then $B'[x_1] = B'[x_2]$, so, yet again by Rolle's Theorem, there must be a number x_3 between x_1 and x_2 , such that $B''[x_3] = 0$. Noting that $0 < x_1 < x_2 < 360$, we see that $x_3 \in (0, 360)$.

Problem 4

■ **a.**

The velocity vector is $\langle x'[t], y'[t] \rangle = \langle 6e^{3t} - 7e^{-7t}, 9e^{3t} + 2e^{-2t} \rangle$. At $t = 0$, this is $\langle -1, 11 \rangle$, and this vector has length $\sqrt{(-1)^2 + (11)^2} = \sqrt{122}$, which is speed at $t = 0$.

■ **b.**

$$\frac{dy}{dx} = \frac{y'[t]}{x'[t]} = \frac{9e^{3t} + 2e^{-2t}}{6e^{3t} - 7e^{-7t}} = \frac{9 + 2e^{-5t}}{6 - 7e^{-10t}} \rightarrow \frac{3}{2} \text{ as } t \rightarrow \infty.$$

■ **c.**

The tangent line is horizontal when $\frac{dy}{dx} = 0$. By part b, above, the tangent line can be horizontal only when $9e^{3t} + 2e^{-2t} = 0$. Because both exponentials are always positive, there are no horizontal tangent lines.

■ **d.**

The tangent line is vertical precisely when the tangent vector is vertical. By part a, the tangent line can be vertical only when $6e^{3t} - 7e^{-7t} = 0$. This is equivalent to $e^{-10t} = \frac{6}{7}$, or $t = -\frac{1}{10} \ln \frac{6}{7}$. (It should be noted that we saw in part c., above, that $y'[t]$ never takes on the value zero; if $y'[-\frac{1}{10} \ln \frac{6}{7}]$ were zero, this would complicate our analysis.)

Problem 5

■ a.

$g[3]$ is the area of a rectangle of base 1 and height 2 plus the area of a triangle of base 1 and height 2, or $2 + 1 = 3$. By the Fundamental Theorem of Calculus, $g'[x] = f[x]$, and when $x = 3$, this is $f[3] = 2$. In the interval $(2, 4)$, the FTC tells us that $g'[x] = f[x] = 8 - 2x$. Thus $g''[x] = -2$ in that interval, and $g''[3] = -2$.

■ b.

The average rate of change of g on $[0, 3]$ is $\frac{1}{3} \int_0^3 g'[x] dx = \frac{1}{3} (g[3] - g[0]) = \frac{1}{3} (3 - (-4)) = \frac{7}{3}$. (See part a for the calculation of $g[3]$; for $g[0]$, we have $g[0] = \int_2^0 f[t] dt$, which is the negative of the area of a triangle of base 2 and height 4, or -4 .)

■ c.

By the Fundamental Theorem of Calculus, $g'[x] = f[x]$. Thus, on the interval $(0, 3)$, $g'[x]$ takes on its average value, $\frac{7}{3}$, at two different points: $x = \frac{7}{6}$, and $x = \frac{17}{6}$.

■ d.

An inflection point occurs where the monotonicity of the derivative changes from increasing to decreasing or vice versa. There are two such points for $g'[x] = f[x]$. (We know $g'[x] = f[x]$ by the Fundamental Theorem of Calculus.) They are $x = 2$ and $x = 5$.

Problem 6

■ a.

The Taylor series about $x = 2$ for f has general term $a_k(x - 2)^k$, where $a_k = \frac{f^{(k)}[2]}{k!} = \frac{(k+1)!}{k! 3^k} = \frac{k+1}{3^k}$. Through the first four terms, we have $f[x] = 1 + \frac{2}{3}(x - 2) + \frac{1}{3}(x - 2)^2 + \frac{4}{27}(x - 2)^3 + \dots$. The general term is $\frac{k+1}{3^k}(x - 2)^k$.

■ b.

We apply the Ratio Test: $\lim_{k \rightarrow \infty} [(\frac{k+2}{3^{k+1}} |x - 2|^{k+1}) / (\frac{k+1}{3^k} |x - 2|^k)] = \frac{1}{3} |x - 2| \lim_{k \rightarrow \infty} \frac{k+2}{k+1} = \frac{1}{3} |x - 2|$. This limit is less than 1 when $|x - 2| < 3$, so the radius of convergence for this series is 3.

■ c.

We have $g[x] - g[2] = \int_2^x f[t] dt = \int_2^x (1 + \frac{2}{3}(t-2) + \frac{1}{3}(t-2)^2 + \frac{4}{27}(t-2)^3 + \dots) dt$, and we may integrate the series term-by-term (as long as $|x-2| < 3$). Thus, $g[x] = 3 + (x-2) + \frac{1}{3}(x-2)^2 + \frac{1}{3^2}(x-2)^3 + \frac{1}{3^3}(x-2)^4 + \dots$. The general term is $\int_2^x \frac{1}{3^{k-1}}(t-2)^{k-1} dt = \frac{1}{3^{k-1}}(x-2)^k$.

■ d.

When $x = -2$, the general term of the series for $g[x]$ becomes $\frac{(-4)^k}{3^{k-1}} = (-4)(-\frac{4}{3})^{k-1}$. As $k \rightarrow \infty$, this does not converge to zero, so the series of which it is the k -th term cannot converge.