Solutions to the
2004 AP Calculus BC Exam
Free Response Questions

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Problem 1.

We are given the traffic flow, i.e., the rate at which cars pass through an intersection, is

```
F[t_] = 82 + 4 Sin[t / 2]

82 + 4 Sin[t/2]
```

in cars per minute.

a.

The function F gives the rate at which cars pass through the intersection, so the total number of cars that pass through the intersection in the 30-minute period $0 \le t \le 30$ is

```
∫_0^30 F[t] ⅆt // N

2474.0775
```

To the nearest whole number, this is 2474.

b.

```
F'[7] // N

-1.8729134
```

Traffic flow is decreasing at $t = 7$, because $F'[7] < 0$.

c.

The average value, in **cars per minute** of traffic flow over the interval $10 \leq t \leq 15$ is

$$\frac{1}{15-10}\int_{10}^{15} \mathbf{F[t]}\, \mathrm{d}\mathbf{t}$$

$$\frac{1}{5}\left(410 + 8\,\mathrm{Cos}[5] - 8\,\mathrm{Cos}\left[\frac{15}{2}\right]\right)$$

Numerically, in **cars per minute:**

```
N[%]

81.899243
```

d.

The average rate of change of the traffic flow, in **cars per minute per minute**, over $10 \leq t \leq 15$ is $\frac{1}{15-10}\int_{10}^{15} F'[t]\, dt = \frac{1}{5}(F[15] - F[10])$, or

$$\frac{1}{5}\,(\mathbf{F[15] - F[10]})$$

$$\frac{1}{5}\left(-4\,\mathrm{Sin}[5] + 4\,\mathrm{Sin}\left[\frac{15}{2}\right]\right)$$

Numerically, in **cars per minute per minute**:

```
N[%]

1.5175394
```

Problem 2

a.

```
f[x_] = 2 x (1 - x)
```

```
2 (1 - x) x
```

```
g[x_] = 3 (x - 1) √x
```

```
3 (-1 + x) √x
```

The area of the shaded region is

$$\int_0^1 (f[x] - g[x])\, dx$$

$$\frac{17}{15}$$

Numerically:

```
N[%]
```

```
1.1333333
```

b.

The volume of the solid generated by rotating the shaded region about the horizontal line $y = 2$ is

$$\int_0^1 \left(\pi\, (2 - g[x])^2 - \pi\, (2 - f[x])^2\right) dx$$

$$\frac{103\,\pi}{20}$$

Numerically:

```
N[%]

16.179202
```

c.

```
h[x_] = k x (1 - x)

k (1 - x) x
```

The volume of the solid given is $\int_0^1 (h[x] - g[x])^2 \, dx = \int_0^1 \left(3\,(x-1)\sqrt{x} + k\,(x-1)\,x\right)^2 dx$, so the desired equation is $\int_0^1 \left(3\,(x-1)\sqrt{x} + k\,(x-1)\,x\right)^2 dx = 15$.

Problem 3.

We are given $\frac{dx}{\mathrm{dt}} = 3 + \cos t^2$; $x[2] = 1$; $y[2] = 8$.

a.

By the Fundamental Theorem of Calculus, $x[4] = x[2] + \int_2^4 x'[t]\, dt = 1 + \int_2^4 (3 + \cos t^2)\, dt$. Integrating numerically, we obtain

```
1 + NIntegrate[3 + Cos[t^2], {t, 2, 4}]

7.1329989
```

b.

Under the assumption that we can solve the parametric equations locally near $t = 2$ for y as a function of x, the Chain Rule yields $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$, or $\frac{dy}{dx} = \left(\frac{dy}{dt}\right) \big/ \left(\frac{dx}{dt}\right)$. Thus, when $t = 2$ we have $\frac{dy}{dx} = -7/(3 + \cos 4)$. Numerically, this is

$$\texttt{N}\left[\frac{\texttt{-7}}{\texttt{3 + Cos[4]}}\right]$$

```
-2.983349
```

An equation for the line tangent to the curve at ($x[2]$, $y[2]$) is therefore $y = 8 - \frac{7}{3+\cos 4}(x - 1)$.

c.

Speed is $|v[t]| = \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}$. At $t = 2$ this is $\sqrt{(-7)^2 + (3 + \cos 4)^2}$. Numerically:

$$\sqrt{\texttt{(-7)}^{2}\texttt{ + (3 + Cos[4])}^{2}}\texttt{ // Expand}$$

$$\sqrt{58 + 6\,\texttt{Cos[4]} + \texttt{Cos[4]}^{2}}$$

```
N[%]

7.3827765
```

d.

We are given that the slope of the tangent line at ($x[t]$, $y[t]$) is $2t + 1$ when $t \geq 3$. From our observations in part b.), and under the assumptions of that part, we must therefore have $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = (2t + 1)(3 + \cos t^2)$ when $t \geq 3$. Thus, we have $\frac{d^2x}{dt^2} = -2t\sin t^2$ and $\frac{d^2y}{dt^2} = 2(3 + \cos t^2) - 2t(2t + 1)\sin t^2$ when $t \geq 3$. When $t = 4$, this gives the acceleration vector $\langle -8\sin 16,\ 6 + 2\cos 16 - 72\sin 16\rangle$. Numerically:

```
N[{-8 Sin[16], 6 + 2 Cos[16] - 72 Sin[16]}]

{2.3032265, 24.81372}
```

Problem 4

Here we are given: $x^2 + 4y^2 = 7 + 3xy$.

■ a.

From what is given, we find, by implicit differentiation, that $2x + 8yy' = 3y + 3xy'$, so that $(8y - 3x)y' = 3y - 2x$. Dividing by $(8y - 3x)$, we obtain $\frac{dy}{dx} = \frac{3y - 2x}{8y - 3x}$, as required.

■ b. Thanks to Bob Enenstein for pointing out that I misspelled zero as "2" in this one.

If we are to have $y' = 0$, then from part a.) we know that $3y - 2x = 0$, and if $x = 3$ we must have $y = 2$. Substituting $x = 3$ and $y = 2$ into the original equation yields $x^2 + 4y^2 = 3^2 + 4 \cdot 2^2 = 9 + 16 = 25 = 7 + 3 \cdot 3 \cdot 2 = 7 + 3xy$. We conclude that $(3, 2)$ meets our requirements.

■ c.

From part a.) above, we have $(8y - 3x)y' = 3y - 2x$. Hence, $y''(8y - 3x) + y'(8y' - 3) = 3y' - 2$. At $x = 3$, $y = 2$, we have $y' = 0$. Substitution then gives, at $(3, 2)$, $(8 \cdot 2 - 3 \cdot 3)y'' = -2$. It then follows that at $(3, 2)$ we must have $y'' = -2/7 < 0$. We conclude, by the Second Derivative Test, that the curve has a local maximum at $(3, 2)$.

Problem 5

Given: $\frac{dP}{dt} = \frac{P}{5}(1 - \frac{P}{12})$.

■ a.

Equilibrium solutions are $P[t] \equiv 0$ and $P[t] \equiv 12$. For $0 < P < 12$, $P'[t] > 0$, while for $12 < P$, $P'[t] < 0$. Hence, any solution whose initial value is positive will be asymptotic to the equilibrium solution $P[12]$ (we take the liberty of interpreting a horizontal line as its own horizontal asymptote). Both of the required limits are therefore 12.

■ b.

$P[t]$ grows fastest when $P'[t]$ is maximal. Thus, we seek to maximize $P'[t] = \frac{P}{5}(1 - \frac{P}{12}) = \frac{P}{5} - \frac{P^2}{60}$. This maximum occurs either when $P = 3$ or when $\frac{d}{dP}\left(\frac{P}{5} - \frac{P^2}{60}\right) = \frac{1}{5} - \frac{P}{30} = 0$. The latter condition implies that $P = 6$. When $P = 3$, $P'[t] = \frac{3}{5} - \frac{9}{60} = \frac{9}{20}$. When $P = 6$, $P'[t] = \frac{6}{5} - \frac{36}{60} = \frac{3}{5}$. Thus, $P[t]$ grows fastest when $P = 6$.

■ c.

If $Y'[t] = \frac{Y[t]}{5}(1 - \frac{t}{12})$, then $\frac{Y'[t]}{Y[t]} = \frac{1}{5}(1 - \frac{t}{12})$, so $\int_0^t \frac{Y'[\eta]}{Y[\eta]} d\eta = \frac{1}{5}\int_0^t (1 - \frac{\eta}{12}) d\eta$. Thus, $\ln \frac{Y[t]}{Y[0]} = \frac{1}{5}\left(t - \frac{t^2}{24}\right)$, so that $Y[t] = Y[0] \exp\left[\frac{1}{5}\left(t - \frac{t^2}{24}\right)\right] = 3 \exp\left[\frac{1}{5}\left(t - \frac{t^2}{24}\right)\right]$.

■ d.

As $t \to \infty$, $\frac{1}{5}\left(t - \frac{t^2}{24}\right) \to -\infty$, so $\lim_{t\to\infty} Y[t] = 0$.

Problem 6

■ a.

If $f[x] = \sin(5x + \pi/4)$ and $P[x]$ is the third degree Taylor polynomial for f about $x = 0$, then $P[x] = f[0] + f'[0]\,x + \frac{f''[0]}{2}\,x^2 + \frac{f'''[0]}{6}\,x^3$. Now $f'[x] = 5\cos[5x + \pi/4]$, $f''[x] = -25\sin[5x + \pi/4]$, $f'''[x] = -125\cos[5x + \pi/4]$, and, indeed, $f^{(4k+1)}[x] = 5^{4k+1}\cos[5x + \pi/4]$, $f^{(4k+2)}[x] = -5^{4k+2}\sin[5x + \pi/4]$, $f^{(4k+3)}[x] = -5^{4k+3}\cos[5x + \pi/4]$, and $f^{(4k)}[x] = 5^{4k}\sin[5x + \pi/4]$. Thus, $P[x] = \frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}\,x - \frac{25\sqrt{2}}{4}\,x^2 - \frac{125\sqrt{2}}{12}\,x^3$.

■ b.

The coefficient of x^{22} in the Taylor series for f about $x = 0$ is $\frac{f^{(22)}[0]}{22!} = \frac{f^{(4\cdot 5+2)}[0]}{22!} = \frac{-5^{22}\sqrt{2}}{2\cdot 22!}$.

■ c.

The Lagrange Remainder for the third order Taylor series at $x = 0$ has the form $\frac{f^{(4)}[\xi]}{24}\,x^4$, where ξ is some number that lies between 0 and x. Now $f^{(4)}[x] = 5^4\sin[5x + \pi/4]$, Thus, $f[\frac{1}{10}] = P[\frac{1}{10}] + \frac{5^4\sin(5\xi+\pi/4)}{24}\left(\frac{1}{10^4}\right)$ for some $\xi \in (0, \frac{1}{10})$. So $|f[\frac{1}{10}] - P[\frac{1}{10}]| = \left|\frac{5^4}{10^4\cdot 24}\right| \cdot |\sin(5\xi + \pi/4)| \le \frac{1}{2^4\cdot 24} = \frac{1}{384} < \frac{1}{100}$, where we have made use of the fact that $|\sin u| \le 1$ for all real u.

■ d.

We obtain the third degree Taylor polynomial about $x = 0$ for $G[x] = \int_0^x f[t]\,dt$ by integrating, term by term, the second degree Taylor polynomial for f. Thus, the required Taylor polynomial is $\frac{\sqrt{2}}{2}\,x + \frac{5\sqrt{2}}{4}\,x^2 - \frac{25\sqrt{2}}{12}\,x^3$.