Solutions to the
2004 AP Calculus BC Exam (Form B)
Free Response Questions

Louis A. Talman
Department of Mathematical & Computer Sciences
Metropolitan State College of Denver

Problem 1. (Thanks to Bill Kehlenbeck for pointing out a careless error that rendered the first three parts of this one wrong.)

■ a.

```
vx[t_] = √(t^4 + 9)
```

$\sqrt{9+t^4}$

```
vy[t_] = 2 ⅇ^t + 5 ⅇ^-t
```

$5\, e^{-t} + 2\, e^{t}$

Speed:

```
s[t_] = √(vx[t]^2 + vy[t]^2)
```

$\sqrt{9 + (5\, e^{-t} + 2\, e^{t})^2 + t^4}$

```
s[0]
```

$\sqrt{58}$

Numerically,

```
N[s[0]]

7.6157731
```

Acceleration:

```
a[t_] = {vx'[t], vy'[t]}
```

$$\left\{\frac{2\,t^3}{\sqrt{9+t^4}},\ -5\,e^{-t}+2\,e^{t}\right\}$$

```
a[0]

{0, -3}
```

b.

The tangent vector is

```
T[t_] = {vx[t], vy[t]}
```

$$\left\{\sqrt{9+t^4},\ 5\,e^{-t}+2\,e^{t}\right\}$$

At time $t = 0$,

```
T[0]

{3, 7}
```

The slope of a line parallel to this vector is $\frac{7}{3}$, so an equation of the line tangent to the curve at $(4,\ 1)$, the point corresponding to $t = 0$, is $y = 1 + \frac{7}{3}(x - 4)$.

c.

Total distance travelled over the interval $0 \le t \le 3$ is $\int_0^3 \sqrt{v_x[t]^2 + v_y[t]^2}\, dt$. Integrating numerically, we obtain

```
NIntegrate[Sqrt[vx[t]^2 + vy[t]^2], {t, 0, 3}]

45.226818
```

d.

The x-coordinate of the particle at time $t = 3$ is $x[3] = x[0] + \int_0^3 x'[t]\, dt = 4 + \int_0^3 \sqrt{t^4 + 9}\, dt$. Numeric integration gives

```
4 + NIntegrate[Sqrt[t^4 + 9], {t, 0, 3}]

17.930791
```

Problem 2

We are given $T[x] = 7 - 9(x-2)^2 - 3(x-2)^3$ as the third-degree Taylor polynomial for a certain function f about $x = 2$.

a.

In any Taylor polynomial, the coefficient a_k of $(x-a)^k$ is $f^{(k)}[a]/k!$. Hence $f[2] = 7$ and $f''[2] = 2 \cdot (-9) = -18$.

b.

Reasoning as in part a.), we find that $f'[2] = 0$, so that f has a critical point at $x = 2$. Because $f''[2] = -18$, the Second Derivative Test allows us to conclude that f has a local maximum at $x = 2$.

c.

$f[0] \simeq 7 - 9 \cdot (-2)^2 - 3 \cdot (-2)^3 = -5$. There is not enough information to determine whether or not f has a critical point at $x = 0$. This is because the third-degree Taylor polynomial carries no information about derivatives at any point other than the point about which the expansion has been done; it is determined solely by the values of the function and its first three derivatives at that point.

d.

The Lagrange Remainder for the third-degree Taylor polynomial at $x = 2$ has the form $\frac{f^{(4)}[\xi]}{4!}(x-2)^4$, where ξ is some unknown number in the interval between x and 2. Thus, $f[0] = T[0] + \frac{1}{24} f^{(4)}[\xi](0-2)^4$ for a certain $\xi \in (0, 2)$. Because $| f^{(4)}[x] | \leq 6$ for all $x \in [0, 2]$, this means that $| f[0] - T[0] | \leq \frac{6}{24}(2)^4 = 4$. But, from part c.) above, we know that $T[0] = -5$. Hence, $-4 \leq f[0] - (-5) \leq 4$, whence $-9 \leq f[0] \leq -1$.

Problem 3.

We are given

```
Inner[Set, Map[v, {0, 5, 10, 15, 20, 25, 30, 35, 40}],
 {7.0, 9.2, 9.5, 7.0, 4.5, 2.4, 2.4, 4.3, 7.3}, List]

{7., 9.2, 9.5, 7., 4.5, 2.4, 2.4, 4.3, 7.3}
```

(This arcane syntax assigns the correct values to v[0], v[5], etc.)

a.

The Mid-Point Rule with four subintervals of equal length gives for $\int_0^{40} v[t]\, dt$ the approximate value

$$\sum_{k=1}^{4} 10\, v[5 + 10\,(k - 1)]$$

```
229.
```

$v[t]$ is given in miles per minute, so the integral gives miles travelled during the time interval over which the integral is taken.

b.

By Rolle's Theorem, acceleration--which is $v'[t]$--must be zero at least once in the interval $0 \leq t \leq 15$, because $v[0] = v[15]$. Similarly, $v'[t]$ must be zero at least once in the interval $25 \leq t \leq 30$, because $v[25] = v[30]$. Thus, acceleration must vanish at least twice in the interval $0 \leq t \leq 40$.

c.

```
f[t_] = 6 + Cos[t/10] + 3 Sin[7 t/40]
```

$$6 + \text{Cos}\left[\frac{t}{10}\right] + 3\,\text{Sin}\left[\frac{7\,t}{40}\right]$$

If the function f models velocity, then acceleration is f'. Thus

```
f'[t]
```

$$\frac{21}{40}\,\text{Cos}\left[\frac{7\,t}{40}\right] - \frac{1}{10}\,\text{Sin}\left[\frac{t}{10}\right]$$

```
% /. t → 23.0
```

−0.40769419

At time $t = 23$, acceleration is -0.408 miles / minute^2.

d.

Average velocity over $0 \le t \le 40$ is $\frac{1}{40}\int_0^{40} f[t]\,dt$:

$$\frac{1}{40}\int_0^{40} f[t]\,dt$$

$$\frac{1}{40}\left(240 + \frac{240}{7}\,\text{Sin}\left[\frac{7}{2}\right]^2 + 10\,\text{Sin}[4]\right)$$

```
N[%]
```

5.9162698

Problem 4:

a.

Inflection points are to be found where f' has relative extrema. Consequently, the function f whose derivative is pictured has inflection points at $x = 1$ and at $x = 3$.

b. Thanks to Tammy Brown, Becky Myers, and Sue Wall, who all pointed out that I'd blown this one completely.

The function f is decreasing on the interval $[-1, 4]$ and increasing on the interval $[4, 5]$ because f' is non-positive on the first of these intervals, and non-negative on the second. Consequently, f takes on its absolute minimum value for the interval $[-1, 5]$ at $x = 4$.

The function f has its absolute maximum at one of the points where $x = -1$ or $x = 5$. (There can be no absolute maximum for f at any point interior to $(-1, 5)$ because f' has in that interval no zeroes at which it undergoes a sign change from positive to negative as x increases.) The area bounded by f and the x-axis on the interval $[-1, 4]$ is clearly larger than the area bounded by f and the x-axis on the interval $[4, 5]$, so $\int_4^{-1} f'[t]\,dt = f[-1] - f[4] > f[5] - f[4] = \int_4^5 f'[t]\,dt$. This means that $f[-1] > f[5]$, so that the absolute maximum value taken on by f in the interval $[-1, 5]$ is $f[-1]$.

c.

We are given $g[x] = x\,f[x]$, so $g'[2] = f[2] + 2 \cdot f'[2] = 6 + 2 \cdot (-1) = 4$. Because $g[2] = 2\,f[2] = 12$, this means that an equation for the line tangent to the graph of g at $x = 2$ is $y = 12 + 4\,(x - 2)$.

Problem 5

a.

The average value of $g[x] = \frac{1}{\sqrt{x}}$ on the interval $[1, 4]$ is $\frac{1}{4-1} \int_1^4 \frac{dx}{\sqrt{x}} = \frac{2}{3}\,x^{1/2} \Big|_1^4 = \frac{2}{3}$.

b.

The volume of the solid generated when the region bounded by the graph of $y = g[x]$, the vertical lines $x = 1$ and $x = 4$, and the x-axis is revolved about the x-axis is $\pi \int_1^4 \frac{1}{x}\,dx = \pi \ln 4$.

c.

The average value of the areas of the cross sections perpendicular to the x-axis is $\frac{\pi}{4-1}\int_1^4 \frac{1}{x}\,dx = \frac{\pi}{3}\ln 4$.

d.

The improper integral $\int_4^\infty g[x]\,dx$ is $\lim_{b\to\infty}\int_4^b \frac{dx}{\sqrt{x}} = \lim_{b\to\infty} 2\,x^{1/2}\Big|_4^b = \lim_{b\to\infty}(2\,b^{1/2}-4) = \infty$, so the improper integral diverges. However, $\lim_{b\to\infty}\frac{1}{b-a}\int_4^b \frac{dx}{\sqrt{x}} = \lim_{b\to\infty}\frac{2\,b^{1/2}-4}{b-4} = 2\lim_{b\to\infty}\frac{(b^{1/2}-2)}{(b^{1/2}-2)\,(b^{1/2}+2)} = 2\lim_{b\to\infty}\frac{1}{b^{1/2}+2} = 0$.

Problem 6

a.

We have $\int_0^1 x^n\,dx = \frac{x^{n+1}}{n+1}\Big|_0^1 = \frac{1}{n+1}$.

b.

If $y = x^n$, then $y' = n\,x^{n-1}$, so that the equation of the line ℓ is $y = 1 + n(x-1)$. This line meets the x-axis when $x = 1 - \frac{1}{n}$, so that the base of the triangle T has length $\frac{1}{n}$. Because the altitude of the triangle T is 1, the area of T is $\frac{1}{2\,n}$.

c.

From what we have seen in parts a.) and b.), the area $A[n]$ of the region S is $A[n] = \frac{1}{n+1} - \frac{1}{2\,n} = \frac{n-1}{2\,n(n+1)}$. Then $A'[n] = -\frac{n^2-2\,n-1}{2\,n^2\,(n+1)^2}$, which is zero for $n > 1$ only when $n = 1+\sqrt{2}$ (by the Quadratic Formula). Noting that $A'[n] > 0$ for $1 \le n < 1+\sqrt{2}$, while $A'[n] < 0$ for $1+\sqrt{2} < n$, we conclude that the maximal area occurs when $n = 1+\sqrt{2}$.