

Solutions to the 2005 AP Calculus BC Exam Free Response Questions

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Problem 1.

■ a)

$$\text{FindRoot}\left[\frac{1}{4} + \text{Sin}[\pi x] == 4^{-x}, \{x, 0\}\right]$$

$$\{x \rightarrow 0.17821805\}$$

$$a = x /. \%$$

$$0.17821805$$

So the required area is

$$\int_0^a \left(4^{-x} - \frac{1}{4} - \text{Sin}[\pi x]\right) dx$$

$$0.064753062$$

■ b)

$$\int_a^1 \left(\frac{1}{4} + \sin[\pi x] - 4^{-x} \right) dx$$

0.41036219

■ c)

$$\pi \int_a^1 \left(\left(\frac{1}{4} + \sin[\pi x] + 1 \right)^2 - (4^{-x} + 1)^2 \right) dx$$

4.5587633

Problem 2.

$$r[\theta] = \theta + \sin[2\theta]$$

$$\theta + \sin[2\theta]$$

■ a)

Area is

$$\frac{1}{2} \int_0^\pi (r[\theta])^2 d\theta$$

$$\frac{1}{2} \left(-\frac{\pi}{2} + \frac{\pi^3}{3} \right)$$

■ b)

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FindRoot[r[θ] Cos[θ] == -2, {θ, 5 π / 6}]
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{θ → 2.7860609}
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■ c)

When $\frac{dr}{d\theta} < 0$, r decreases as θ increases. This means that as θ ranges upward from $\pi/3$ to $2\pi/3$, the corresponding point on the curve gets closer to the pole.

■ d)

We seek to find the absolute maximum value of $r(\theta)$ on the interval $[0, \pi/2]$. This will be found where $\theta = 0$ or $\theta = \pi/2$ or $r'(\theta) = 0$. Now $r(0) = 0$ and $r(\pi/2) = \pi/2 \sim 1.571$. This eliminates $\theta = 0$ from contention. $r'(\theta) = 0$ when $1 + 2 \cos 2\theta = 0$, or when $\theta = \pi/3$. We have $r(\pi/3) = (\sqrt{3}/2) + (\pi/3) \sim 1.913$. We conclude that the maximum distance from the pole to the curve for $0 \leq \theta \leq \pi/2$ occurs when $\theta = \pi/3$.

Problem 3.

■ a)

$$\frac{55 - 62}{8 - 6}$$

$$-\frac{7}{2}$$

Thus $T'(7)$ is approximately $\frac{T(8)-T(6)}{8-6}$ degrees C per centimeter, or $-7/2$ degrees C. per centimeter.

■ b)

Average temperature of the wire is $\frac{1}{8} \int_0^8 T(x) dx$. This is

$$\frac{1}{8} \left(\frac{100 + 93}{2} + \frac{93 + 70}{2} (5 - 1) + \frac{70 + 62}{2} + \frac{62 + 55}{2} (8 - 6) \right)$$

$$\frac{1211}{16}$$

Answer: 1211 / 16 degrees Celsius.

■ c)

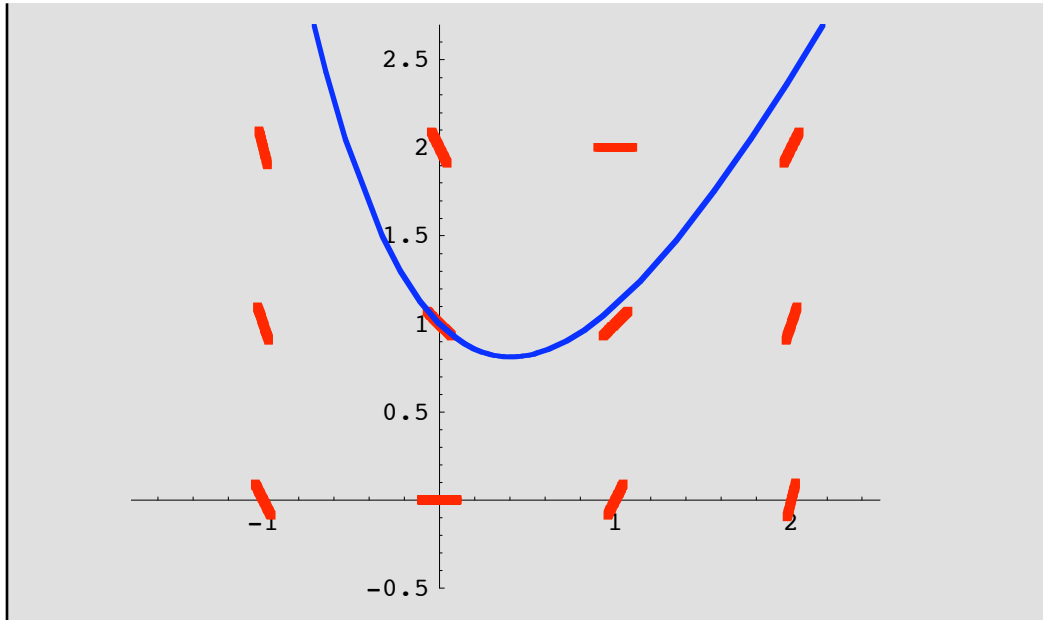
We are given that T is twice differentiable (although we are not told where). This means that T' is continuous--on the interval $[0,8]$, we hope, so that the problem is meaningful. By the Fundamental Theorem of Calculus, $\int_0^8 T'(x) dx = T(8) - T(0) = -45$ degrees Celsius. The integrand $T'(x)$ is the (instantaneous) rate at which $T(x)$ changes per unit length at each point of the interval $[0, 8]$, and the integral gives net temperature change over the interval $[0, 8]$.

■ d)

By the Mean Value Theorem, there is a point $\xi \in [1, 5]$ where $T'(\xi) = [T(5) - T(1)] / (5 - 1) = -23/4$. By the Mean Value Theorem again, there is a point $\eta \in [5, 6]$ where $T'(\eta) = T(6) - T(5) = -8$. Note that, necessarily, $0 < \xi < \eta < 8$. A third application of the Mean Value Theorem---this time to T' ---assures us that there is a point $\zeta \in [\xi, \eta] \subseteq (0, 8)$ such that $T''(\zeta) = [T'(\eta) - T'(\xi)] / (\eta - \xi) = (-8 + 23/4) / (\eta - \xi) = -9 / [4(\eta - \xi)] < 0$. Consequently, these data are not consistent with the assertion that $T''(x) > 0$ for every x in $(0, 8)$.

Problem 4.

■ a)



■ b)

At a local minimum, we must have $y' = 0$, and because we are on a solution curve this tells us that $0 = 2x - y = 2 \ln(3/2) - y$. Hence, the y -coordinate of this minimum is $y = \ln(9/4)$.

■ c)

Because $y(0) = 1$, we have $y'(0) = 2 \cdot 0 - 1 = -1$.

Consequently,

$$y(-0.2) \sim y(0) + y'(0)(-0.2) = 1 - (-0.2) = 1.2.$$

Consequently,

$$y'(-0.2) = 2 \cdot (-0.2) - y(0.2) \sim -0.4 - (1.2) = -1.6.$$

So

$$y(-0.4) \sim y(-0.2) + y'(-0.2)(-0.4 + 0.2) = 1.2 + (-1.6)(-0.2) = 1.52.$$

■ d)

If $y' = 2x - y$, then $y'' = 2 - y' = 2 - (2x - y) = y + 2 - 2x$. This latter quantity is positive when $x \leq 0$ and $y > 0$ —as it surely is for our solution. Hence the solution must be concave upward in the region where we are examining it. Consequently tangent lines to the solution lie below the curve throughout the interval $(-0.4, 0)$. Because Euler's Method proceeds by replacing the curve with tangent lines, the method underestimates y in this region.

Problem 5.

■ a)

$\int_0^{24} v(t) dt = \frac{1}{2} \cdot (4-0) \cdot 20 + (16-4) \cdot 20 + \frac{1}{2} \cdot (24-16) \cdot 20 = 40 + 240 + 80 = 360$ meters. $\int_0^{24} v(t) dt$ gives the distance, in meters, traveled by the car during the time period $0 \leq t \leq 24$.

■ b)

$v'(4) = \lim_{h \rightarrow 0} [v(4+h) - v(4)]/h$ is the definition of $v'(4)$. We note that for $0 < t \leq 4$, we have $v(t) = 5t$, so that $v(4+h) - v(4) = 5(4+h) - 5 \cdot 4 = 5h$ when $h < 0$ but $|h|$ is small. Thus $[v(4+h) - v(4)]/h = 5$ for such values of h . But when $4 \leq t \leq 16$, we have $v(t) = 20$, so that $[v(4+h) - v(4)] = 20 - 20 = 0$ when $h > 0$ and $|h|$ is small. For such h we therefore have $[v(4+h) - v(4)]/h = 0$. Consequently, $\lim_{h \rightarrow 0^+} [v(4+h) - v(4)]/h = 0$, but $\lim_{h \rightarrow 0^-} [v(4+h) - v(4)]/h = 5$. The two one-sided limits have different values, so the two-sided limit does not exist. this means that $v'(4)$ does not exist.

Now $v(t) = 60 - \frac{5}{2}t$ when $16 < t < 24$. Hence $v'(t) = -\frac{5}{2}$ for all t for which $16 < t < 24$, by the standard rule for differentiating polynomials. It follows that $v'(20) = -\frac{5}{2}$.

■ c)

Acceleration is given by $a(t) = 5$ when $0 < t < 4$, by $a(t) = 0$ when $4 < t < 16$, and by $a(t) = -\frac{5}{2}$ when $16 < t < 24$.

■ d)

The average rate of change of v over $8 \leq t \leq 20$ is $[v(20) - v(8)]/(20 - 8) = (10 - 20)/(20 - 8) = -5/6$. The hypotheses of the Mean Value Theorem require that $v'(t)$ exist at every point of the interior of an interval on which we wish to apply the theorem. The Mean Value Theorem is therefore inapplicable to v on $[8, 20]$ because $v'(16)$ does not exist.

Problem 6.

■ a)

The 6-th degree Taylor polynomial about $x = 2$ for this function is

$$f(2) + f'(2)(x-2) + \frac{1}{2!} f''(2)(x-2)^2 + \frac{1}{3!} f^{(3)}(2)(x-2)^3 + \frac{1}{4!} f^{(4)}(2)(x-2)^4 + \frac{1}{5!} f^{(5)}(2)(x-2)^5 + \frac{1}{6!} f^{(6)}(2)(x-2)^6 = 7 + \frac{1}{2 \cdot 3^2} (x-2)^2 + \frac{1}{4 \cdot 3^4} (x-2)^4 + \frac{1}{6 \cdot 3^6} (x-2)^6$$

■ b)

The coefficient of $(x-2)^{2n}$ in the Taylor expansion of this function about $x = 2$ is $1/[(2n) 3^{2n}]$.

■ c)

We have $\lim_{n \rightarrow \infty} \frac{a_{2n+2}|x-2|^{2n+2}}{a_{2n}|x-2|^{2n}} = \lim_{n \rightarrow \infty} \frac{(2n) 3^{2n}|x-2|^{2n+2}}{(2n+2) 3^{2n+2}|x-2|^{2n}} = \lim_{n \rightarrow \infty} \frac{n|x-2|^2}{9(n+1)} = \frac{1}{9} |x-2|^2$. Consequently, the Ratio Test guarantees that the series converges when $|x-2|^2 < 9$, or when $-1 < x < 5$. When $|x-2| = 3$, the general term of the series becomes $(x-2)^{2n}/[(2n) 3^{2n}] = 3^{2n}/[(2n) 3^{2n}] = 1/(2n)$. This is a constant multiple of the harmonic series, and so diverges. The interval of convergence is therefore $-1 < x < 5$.