Solutions to the
2005 AP Calculus AB (Form B) Exam
Free Response Questions

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Problem 1.

a)

The general acceleration vector is:

```
D[{12 t - 3 t^2, Log[1 + (t - 4)^4]}, t]
```

$$\left\{12-6\,t,\ \frac{4\,(-4+t)^3}{1+(-4+t)^4}\right\}$$

When $t = 2$, this gives:

```
% /. t → 2
```

$$\left\{0,\ -\frac{32}{17}\right\}$$

Speed is

```
s[t_] = Sqrt[(12 t - 3 t^2)^2 + (Log[1 + (t - 4)^4])^2]
```

$$\sqrt{(12\,t-3\,t^2)^2+\text{Log}\left[1+(-4+t)^4\right]^2}$$

When $t = 2$, this is

```
s[2]
```

$\sqrt{144 + \text{Log}[17]^2}$

Numerically:

```
N[s[2]]

12.32992692
```

b)

The *y*-coordinate of *P*, the object's position when $t = 2$, is given by $5 + \int_0^2 \ln[1 - (t-4)^4]\, dt$. Integrating numerically, we find

```
5 + NIntegrate[Log[1 + (t - 4)^4], {t, 0, 2}]

13.6714512905
```

c)

The tangent line at t=2 is parallel to the vector $(x'(2),\ y'(2)\} = (12,\ \ln(17))$, and thus has slope $\ln(17)/12$. An equation is therefore $y = 13.671451 + [\ln(17)/12](x - 3)$.

d)

The object is at rest precisely when both x' and y' vanish. Now $x' = 12t - 3t^2 = 0$ when $t = 0$ and when $t = 4$. $y' = \ln[1 + (t-4)^2]$ doesn't vanish when $t = 0$, but does when $t = 4$. Thus, velocity vanishes iff $t = 4$.

Problem 2.

```
W[t_] = 95 Sqrt[t] (Sin[t / 6])^2
```

$95 \sqrt{t}\ \text{Sin}\left[\frac{t}{6}\right]^2$

```
R[t_] = 275 (Sin[t / 3])^2
```

$275\,\text{Sin}\left[\frac{t}{3}\right]^2$

a)

```
W[15] - R[15]
```

$95\sqrt{15}\,\text{Sin}\left[\frac{5}{2}\right]^2 - 275\,\text{Sin}[5]^2$

```
N[%]
```

```
-121.090025015
```

When t = 15, The difference is negative, so water is being removed from the tank at a higher rate than it is being pumped in, so the amount of water in the tank at that time is decreasing.

b)

The amount of water in the tank at time t is $A[t] = 1200 + \int_0^t (W[\tau] - R[\tau])\, d\tau$. Integrating numerically, we find that $A[18]$ is

```
1200 + NIntegrate[(W[τ] - R[τ]), {τ, 0, 18}]
```

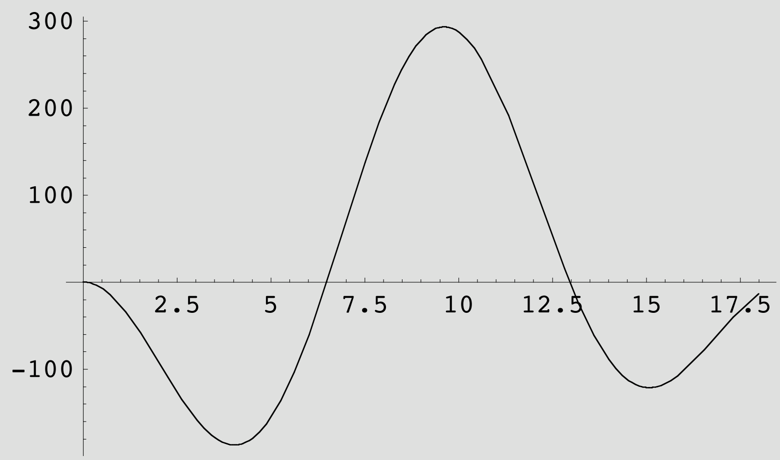
```
1309.78818325
```

To the nearest whole number, this is 1310 gallons.

c)

We have $A'[t] = W(t) - R(t)$. We therefore seek the zeros of $W[t] - R[t]$. From the graph,

```
Plot[W[t] - R[t], {t, 0, 18}]
```




```
- Graphics -
```

we see that these zeros are near $t = 0$, $t = 6.5$, and $t = 13.0$. The first of these is at $t = 0$. We solve numerically for the second and the third.

```
FindRoot[W[t] - R[t] == 0, {t, 6.5}]

{t → 6.49484017118}
```

```
r2 = t /. %

6.49484017118
```

```
FindRoot[W[t] - R[t] == 0, {t, 13}]

{t → 12.9748246724}
```

```
r3 = t /. %

12.9748246724
```

We calculate A for each of these and for $t = 18$. First we note that $A[0] = 1200$.

```
A[t_] := 1200 + NIntegrate[(W[τ] - R[τ]) , {τ, 0, t}]
```

```
A[r2]

525.242152722
```

```
A[r3]

1697.44123623
```

```
A[18]

1309.78818325
```

The amount of water in the tank is minimal when $t \sim 6.495$.

d)

With A and R defined as above, we must solve $A(18) - \int_{18}^{k} R(\tau)\, d\tau = 0$ for k.

Problem 3.

a)

At $x = 0$, we have $f'(0) = 0$ because the tangent line is given horizontal. From the formula given for the derivatives, we find that $f''(0) = -3/25$, and the Second Derivative Test now assures us that f has a local maximum at $x = 0$.

b)

The third-degree Taylor polynomial for f at $x = 0$ is $f(0) + f'(0)\,x + \frac{1}{2} f''(0)\,x^2 + \frac{1}{6} f'''(0)\,x^3$, or $6 - (3/25)\,x^2 + (1/125)\,x^3$.

c)

We consider first the magnitudes of the coeffients in the Taylor series:

```
a[n_] = (n + 1)/(5^n (n - 1)^2)
```

$$\frac{5^{-n}\,(1+n)}{(-1+n)^2}$$

We have

```
Limit[a[n + 1] / a[n], n → Infinity]
```

$$\frac{1}{5}$$

Consequently, the series converges for $|x| < 5$. When $x = 5$, the general term of the series becomes $(-1)^{n+1}\,(n+1)/(n-1)^2$. Putting $g(x) = (x+1)/(x-1)^2$, we find that

```
g[x_] = (x + 1)/(x - 1)^2
```

$$\frac{1+x}{(-1+x)^2}$$

```
g'[x] // Together
```

$$\frac{-3-x}{(-1+x)^3}$$

Thus, $g'(x) < 0$ for $x > 1$, so that the sequence a_n is a decreasing sequence for large n. Because $\lim_{n\to\infty} a_n = 0$, we conclude by the Alternating Series Test that the series converges for $x = 5$.

When $x = -5$, the general term of the series is $-(n+1)/(n-1)^2$. We recall that the harmonic series diverges, and we note that

```
Limit[(1 / n) / ((n + 1) / (n - 1)^2), n → Infinity]
```

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thus, the series diverges by the Comparison Limit Test.

The desired interval of convergence is therefore the interval $(-5, 5]$.

Problem 4.

■ a)

$g(-1) = \int_{-4}^{x} f(t)\,dt$ is the negative of the area of the trapezoid defined by the x-axis, the vertical lines $x = -4$ and $x = -1$ and the line segment joining the points $(-4, -3)$ and $(-1, -2)$. Thus, $g(-1) = \frac{-1}{2}(3+2)\cdot 3 = \frac{-15}{2}$. By the Fundamental Theorem of Calculus, $g'(x) = f(x)$, so $f'(-1) = -2$. Because $g'(x) = f(x)$, it follows that $g''(x) = f'(x)$, iff the latter exists. Because of the corner in the graph of $f(x)$ at the point corresponding to $x = -1$, $f'(-1)$ does not exist. (In fact, $f_{-}'(-1) = 1/3$, while $f_{+}'(-1) = 2$.) Thus, $g''(-1)$ does not exist.

■ b)

The inflection points of g occur at the values of x for which $g' = f$ has relative extrema. But f has just one relative extremum in the interval $(-4, 3)$, at $x = 1$--as is evident from the graph. Thus the only relative extremum for g is to be found at $x = 1$.

■ c)

If $h(x) = \int_{x}^{3} f(t)\,dt = -\int_{3}^{x} f(t)\,dt$, then the zeros of h are to be found at those values of x for which the graph of f has just as much area above the x-axis as below in the interval $[x, 3]$. These values are evidently $x = -1$ an $x = 1$.

■ d)

With h given as in part c), above, we have by the Fundamental Theorem of Calculus $h'(x) = -f(x)$. Therefore, h is decreasing on the closures of those intervals for which $-f(x) < 0$, or, equivalently, where $f'(x) > 0$. From the graph, it is evident that h is decreasing on $[0, 2]$.

Problem 5.

■ a)

By implicit differentiation, we have $2yy' = y + xy'$, so that $(2y - x)y' = y$, or $y' = y/(2y - x)$, provided that $2y - x \neq 0$. But $2y - x = 0$ implies that $x = 2y$, and, substituting this in the original equation, we find that $y^2 = 2 + (2y)y$, or $y^2 + 2 = 0$. This is not possible for real y, so we conclude that $y' = y/(2y - x)$ whenever (x, y) lies on the curve $y^2 = 2 + xy$.

■ b)

If $y' = 1/2$, then $1/2 = y/(2y - x)$, or $2y - x = 2y$. Thus $x = 0$. Now if (x, y) lies on the curve $y^2 = 2 + xy$ and $x = 0$, then $y^2 = 2$. The required points are thus $\left(0, \sqrt{2}\right)$ and $\left(0, -\sqrt{2}\right)$.

■ c)

If the tangent line to $y^2 = 2 + xy$ is horizontal at a point (x_0, y_0), then we must have $0 = y'(x_0) = y_0/(2y_0 - x_0)$, and this implies that $y_0 = 0$. But then $0 = y_0^2 = 2 + x_0 y_0 = 2 + 0 = 2$, or $0 = 2$. The contradiction shows that there can be no point on the curve $y^2 = 2 + xy$ where the tangent line is horizontal.

■ d)

We differentiate the equation for the curve implicitly again, but this time we treat x and y both as functions of t and the prime means differentiation with respect to t. This gives $2yy' = x'y + xy'$. Putting $y = 3$, $y' = 6$ in both the original equation and the derived equation leads to the system $9 = 2 + 3x$; $36 = 3x' + 6x$. From the first of these two, we find that $x = 7/3$. Substituting this for x in the second equation, we find that $36 = 3x' + 14$, whence $x' = 22/3$.

Problem 6.

■ a)

The required area is $\int_0^k \frac{dx}{x+2} = \ln(k+2) - \ln 2$.

■ b)

The required volume is $\pi \int_0^k \frac{dx}{(x+2)^2} = \frac{k\pi}{2(k+2)}$.

■ c)

The volume of the solid S is given by the improper integral $\pi \int_k^\infty \frac{dx}{(x+2)^2} = \lim_{T\to\infty} \int_k^T \frac{dx}{(x+2)^2} = \lim_{T\to\infty} \frac{\pi(T-k)}{(k+2)(T+2)} = \frac{\pi}{k+2}$. Hence we must solve $\frac{\pi}{k+2} = \frac{k\pi}{2(k+2)}$ for k. This results in $k = 2$.