Solutions to the
2006 AP Calculus BC Exam
Free Response Questions

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Part A

Problem 1.

■ a)

The curves intersect where $\ln x = x - 2$.

```
In[1]:= a = x /. FindRoot[Log[x] == x - 2, {x, 0.1}][[1]]

Out[1]= 0.158594339563

In[2]:= b = x /. FindRoot[Log[x] == x - 2, {x, 1.6}][[1]]

Out[2]= 3.14619322062
```

In[3]:= $\int_a^b (\text{Log}[x] - (x - 2))\, dx$

```
Out[3]= 1.94909092741
```

- **Answer: 1.949**

- **b)**

```
In[4]:= π ∫_a^b ((Log[x] - (- 3))^2 - ((x - 2) - (-3))^2) ⅆx
```

```
Out[4]= 34.198613526
```

- **Answer: 34.199**

- **c)**

- **Answer: Either of the integrals $\pi \int_{a-2}^{b-2} [(y+2)^2 - e^{2y}]\, dy$ or $2\pi \int_a^b x\,[\ln x - (x-2)]\, dx$ will do. Evaluation was not required; however**

```
In[5]:= 2 π ∫_a^b x (Log[x] - (x - 2)) ⅆx
```

```
Out[5]= 17.099306763
```

and, of course,

```
In[6]:= π ∫_(a-2)^(b-2) ((y + 2)^2 - ⅇ^(2 y)) ⅆy
```

```
Out[6]= 17.099306763
```

Problem 2:

- **a)**

The rate at which left turns happen is:

```
In[7]:= L[t_] = 60 √t (Sin[t/3])^2
```

```
Out[7]= 60 √t Sin[t/3]^2
```

The number of turns when $0 \le t \le 18$ is therefore

```
In[8]:= NIntegrate[L[t], {t, 0, 18}]
```

```
Out[8]= 1657.82373452
```

- **Answer: To the nearest whole number, this is 1658.**

■ b)

From the graph, we see that $L[t] \geq 150$ on $[a, b]$ where $a \sim 12$ and $b \sim 16$. Solving numerically for a and b, we find

```
In[9]:= a = t /. FindRoot[L[t] == 150, {t, 12}][[1]]

Out[9]= 12.4283095349

In[10]:= b = t /. FindRoot[L[t] == 150, {t, 16}][[1]]

Out[10]= 16.1216568612
```

In[11]:= $\frac{1}{b-a}\int_a^b L[t]\,dt$

```
Out[11]= 199.426116196
```

- **Answer: There are 150 or more left turns per hour approximately when $12.428 \leq t \leq 16.122$, where t is measured in hours. The average during this interval is 199.426 left turns per hour.**

■ c)

During the two-hour interval $13 \leq t \leq 15$,

```
In[12]:= NIntegrate[L[t], {t, 13, 15}]

Out[12]= 431.931400444
```

cars make left turns. 500 oncoming cars pass straight through the intersection in this two-hour period. The product of these two numbers is 215,967.7, and this exceeds the threshhold of 200, 000.

- **Answer: The intersection requires a traffic signal. The reasoning is given in the preceding paragraph.**

Problem 3:

■ a)

The velocity vector at time t is $\{\sin^{-1}(1 - 2e^{-t}), \frac{4t}{1+t^3}\}$, so the acceleration vector is

```
In[13]:= D[{ArcSin[1 - 2 ⅇ^-t], 4 t/(1 + t^3)}, t] // Together
```

$$\text{Out[13]= } \left\{\frac{e^{-t}}{\sqrt{e^{-2t}\,(-1+e^{t})}},\ -\frac{4\,(-1+2\,t^3)}{(1+t^3)^2}\right\}$$

When $t = 2$, this is

```
In[14]:= % /. t → 2
```

$$\text{Out[14]= } \left\{\frac{1}{\sqrt{-1+e^2}},\ -\frac{20}{27}\right\}$$

Speed is $\sqrt{v \cdot v}$, so

```
In[15]:= s[t_] = Sqrt[{ArcSin[1 - 2 ⅇ^-t], 4 t/(1 + t^3)}.{ArcSin[1 - 2 ⅇ^-t], 4 t/(1 + t^3)}]
```

$$\text{Out[15]= } \sqrt{\frac{16\,t^2}{(1+t^3)^2} + \text{ArcSin}[1 - 2\,e^{-t}]^2}$$

```
In[16]:= s[2]
```

$$\text{Out[16]= } \sqrt{\frac{64}{81} + \text{ArcSin}\left[1 - \frac{2}{e^2}\right]^2}$$

- **Answer: When $t = 2$, the acceleration vector is $\left\langle \frac{1}{\sqrt{e^2-1}}, -\frac{20}{27} \right\rangle$, and speed is $\sqrt{\frac{64}{81} + \left[\sin^{-1}\left(1 - \frac{2}{e^2}\right)\right]^2}$.**

- **b)**

The curve can have a vertical tangent only when $\frac{dx}{dt} = 0$, This condition is met just when $1 - 2e^{-t} = 0$, or when $t = \ln 2$.

- **Answer: $t = \ln 2$.**

- **c)**

$m(t) = y'(t)/x'(t) = 4\,t \sin^{-1}(1 - 2\,e^{-t})/(1 + t^3)$. We have $\lim_{t\to\infty} m(t) = \lim_{t\to\infty} 4\,t \sin^{-1}(1 - 2\,e^{-t})/(1 + t^3)$. But as $t \to \infty$, $\sin^{-1}(1 - 2\,e^{-t}) \to \pi/2$. On the other hand, $\lim_{t-\infty} 4\,t/(1 + t^3) = 0$. Hence $\lim_{t\to\infty} m(t) = 0$.

- **Answer: $\lim_{t \to \infty} m(t) = 0$.**

- **d)**

By the Fundamental Theorem of Calculus, $x(t) = x(2) + \int_2^t x'(\tau)\, d\tau = 6 + \int_2^t \arcsin(1 - 2e^{-\tau})\, d\tau$ and $y(t) = y(2) + \int_2^t y'(\tau)\, d\tau = -3 + \int_2^t [4\tau/(1 + \tau^3)]\, d\tau$. If $y = c$ is a horizontal asymptote for the curve, then we must have both $\lim_{t \to a} x(t) = \pm\infty$ and $\lim_{t \to a} y(t) = c$ for some $a > 2$—and it is possible that we may want $a = \infty$. Let us note that $\tau \geq 2$ implies that $x'(\tau) \geq \arcsin(1 - 2e^{-2}) \geq 0.81$. Consequently, $\int_2^t x'(\tau)\, d\tau \to \infty$ as $t \to \infty$. But $\int_2^t x'(\tau)\, d\tau$ is finite for all real numbers $t \geq 2$. Hence, $x(t) \to \infty$ only as $t \to \infty$. So the horizontal asymptote must lie at $y = -3 + \int_2^\infty [4\tau/(1 + \tau^3)]\, d\tau$. (Note: This integral converges because $0 < 4t/(1 + t^3) = 4/[(1/t) + t^2] < 4/t^2$ when $t \geq 2$, and $\int_2^\infty (4/\tau^2)\, d\tau$ is a convergent improper integral.)

[Special thanks to Jennifer Nichols, of Walnut High School, who pointed out the flawed thinking in my first attempt at this one.]

- **Answer:** $-3 + \int_2^\infty [4\tau/(1+\tau^3)]\,d\tau$.

Part B:

Problem 4:

- **a)**

- **Answer:** Average acceleration is $\frac{v(80)-v(0)}{80-0} = \frac{49-5}{80} = \frac{11}{20}$ feet per second per second.

- **b)**

- **Answer:** $\int_{10}^{70} v(t)\,dt$ measures the distance from the rocket's position at time $t = 10$ to its position at time $t = 70$. The midpoint Riemann sum with three subdivisions of equal length is $v(20)\cdot 20 + v(40)\cdot 20 + v(60)\cdot 20 = 440 + 700 + 880 = 2020$ feet.

- **c)**

- **Answer:** For rocket B, we have, by the Fundamental Theorem of Calculus, $v(80) = v(0) + \int_0^{80} a(\tau)\,d\tau = 2 + 3\int_0^{80} \frac{d\tau}{\sqrt{\tau+1}} = 50$ ft/sec. Thus Rocket A is traveling at 49 ft/sec when $t = 80$, so rocket B is traveling faster.

Problem 5:

- **a)**

- **Answer:** When $x = -1$ and $y = -4$, $\frac{dy}{dx} = 5x^2 - \frac{6}{y-2} = \left[5(-1)^2 - \frac{6}{-4-2}\right] = 6$. At the same point, we have $\frac{d^2y}{dx^2} = 10x + \frac{6y'}{(y-2)^2} = 10(-1) + \frac{6\cdot 6}{(-4-2)^2} = -9$.

- **b)**

- **Answer: If the solution to this initial value problem is tangent to the x-axis at some point, then there is x_0 for which $f(x_0) = 0$ and $f'(x_0) = 0$. This would mean that $0 = f'(x_0) = 5x_0^2 - \frac{6}{0-2} = 5x_0^2 + 3$, which is not possible because the latter quantity must be positive. Thus, the graph of the function f cannot have the x-axis as a tangent line at any point.**

c)

The second-degree Taylor polynomial for f in powers of $(x+1)$ is $f(-1) + f'(-1)(x+1) + \frac{1}{2} f'(-1)(x+1)^2$, or, using the results of Part a) above,

- **Answer: $-4 + 6(x+1) - \frac{9}{2}(x+1)^2$**

d)

Euler's method for this equation is $x_k = x_{k-1} + h$; $y_k = y_{k-1} + f'(x_{k-1})h$, where $h = \frac{1}{2}$, $x_0 = -1$, and $y_0 = -4$. Thus, $x_1 = -\frac{1}{2}$, $y_1 = -4 + 6 \cdot \frac{1}{2} = -1$, $x_2 = 0$, $y_2 = -1 + [5(-1/2)^2 - 6/(-1-2)] \cdot (1/2) = 5/8$.

- **Answer: $f(0) \sim \frac{5}{8}$.**

Problem 6:

a)

We have $\lim_{n\to\infty} [(\frac{n+1}{n+2} |x|^{n+1})/(\frac{n}{n+1} |x|^n)] = |x| \lim_{n\to\infty} \frac{(n+1)^2}{n(n+2)} = |x|$. This is less than one when $|x| < 1$, so by the Ratio Test, the interior of the interval of convergence for this series is $(-1, 1)$. When $|x| = 1$, the magnitude of the n-th term of the series is $\frac{n}{n+1}$, which goes to 1 as $n \to \infty$. The series therefore diverges when $|x| = 1$ for its terms do not go to zero in the limit as $n \to \infty$.

- **Answer: The interval of convergence of the series for f is (-1, 1).**

b)

In order to be sure that $y'(0)$ and $y''(0)$ are meaningful, we must know that the power series for g converges on some non-degenerate interval centered at $x = 0$. (We have shown that a similar statement is true for f in Part a) above.) The Ratio Test easily establishes that, in fact, the series in question converges for all x: We have $\lim_{n\to\infty} ([|x|^{n+1}/(2n+2)!]/[|x|^n/(2n)!]) = |x| \lim_{n\to\infty} (1/[(2n+2)(2n+1)]) = 0$. Thus, we may write $f(x) - g(x) = (\frac{-x}{2} + \frac{2}{3}x^2 + R(x)) - \left(1 - \frac{x}{2!} + \frac{x^2}{4!} + Q(x)\right)$, where $R(x)$ and $Q(x)$ represent terms of order greater than or equal to 3. Hence $H(x) = f(x) - g(x) = -1 + 0x + \frac{5}{8}x^2 + T(x)$, where $T(x)$ represents terms of higher order. It follows that $H'(0) = 0$ and that $H''(0)/(2!) = 5/8$. Thus, $y'(0) = 0$, and $y''(0) = \frac{5}{4}$.

- **Answer:** **$y'(0) = 0$; $y''(0) = \frac{5}{4}$. By the Second Derivative Test, y has a relative minimum at $x = 0$.**