Solutions to the 2006 AP Calculus BC Exam (Form B) Free Response Questions

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Part A

Problem 1:

■ a)

We must first find the intersection of the curve $y = f(x)$ with the x-axis:

In[1]:= $f[x\_] = \frac{x^3}{4} - \frac{x^2}{3} - \frac{x}{2} + 3\,\mathrm{Cos}[x]$

Out[1]= $-\frac{x}{2} - \frac{x^2}{3} + \frac{x^3}{4} + 3\,\mathrm{Cos}[x]$

```
In[2]:= a = x /. FindRoot[f[x] == 0, {x, -1.3}]

Out[2]= -1.37312166468
```

The area of the region R is then

In[3]:= $\int_a^0 f[x]\,dx$

Out[3]= 2.9030944245

- **Answer: To three decimal places, this is 2.903.**

- **b)**

Using the method of washers, the required volume is

```
In[4]:=  π ∫_a^0 ((f[x] - (-2))^2 - (0 - (-2))^2) dx
```

$$\pi \int_a^0 \left((f[x] - (-2))^2 - (0 - (-2))^2 \right) dx$$

```
Out[4]=  59.3614055798
```

- **Answer: 59.361**

- **c)**

We must find the equation of the line tangent to the curve $y = f(x)$ at $(0, 3)$:

```
In[5]:=  f'[x]
```

Out[5]= $-\frac{1}{2} - \frac{2\,x}{3} + \frac{3\,x^2}{4} - 3\,\mathrm{Sin}[x]$

```
In[6]:=  % /. x → 0
```

Out[6]= $-\frac{1}{2}$

The equation of the tangent line is $y = 3 - x/2$. This line meets the curve when x is slightly bigger than 3:

```
In[7]:=  b = x /. FindRoot[f[x] == 3 - x/2, {x, 3.0}]
```

```
Out[7]=  3.38986766308
```

- **Answer: The required integral is $\int_0^b \left[3 - \frac{x}{2} - f(x)\right] dx$.**

Evaluation is not required. Nevertheless:

```
In[8]:=  ∫_0^b (3 - x/2 - f[x]) dx
```

$$\int_0^b \left(3 - \frac{x}{2} - f[x]\right) dx$$

```
Out[8]=  6.98199992437
```

Problem 2:

■ a)

At any point $(x(t_0),\ y(t_0))$, the slope of the tangent line to the solution is $y'(t_0)/x'(t_0)$. Thus, the slope of the tangent to the solution at $(2, -3)$ is $\sec(e^{-1})/\tan(e^{-1}) = \csc(e^{-1})$, and the equation of the tangent line at $(2, -3)$ is $y = -3 + (x - 2)\csc(e^{-1})$.

■ **Answer:** $y = -3 + (x - 2)\csc(e^{-1})$**.**

■ b)

The acceleration vector is $\frac{d}{dt}\,\boldsymbol{v}(t) = \frac{d}{dt}\langle\tan(e^{-t}),\ \sec(e^{-t})\rangle = \langle -e^{-t}\sec^2(e^{-t}),\ -e^{-t}\sec(e^{-t})\tan(e^{-t})\rangle$. Speed is $\sqrt{\boldsymbol{v}(t)\cdot\boldsymbol{v}(t)} = \sqrt{\tan^2(e^{-t}) + \sec^2(e^{-t})}$. Thus,

■ **Answer: When $t = 1$, acceleration is $-e^{-1}\langle\sec^2(e^{-1}),\ \sec(e^{-1})\tan(e^{-1})\rangle$ and speed is $\sqrt{\tan^2(e^{-1}) + \sec^2(e^{-1})}$.**

■ c)

Total distance traveled over the time interval $1 \le t \le 2$ is $\int_1^2 \sqrt{\tan^2(e^{-t}) + \sec^2(e^{-t})}\, dt$.

```
In[9]:= NIntegrate[√((Tan[e^-t])^2 + (Sec[e^-t])^2), {t, 1, 2}]

Out[9]= 1.05934519761
```

- **Answer: Total distance traveled when $1 \le t \le 2$ is, to three decimal places, 1.059.**

- **d)**

- **Answer: If $t \ge 0$, then $0 < e^{-t} \le 1 < \pi/2$. Thus $\frac{dx}{dt} = \tan(e^{-t}) > 0$ when $t \ge 0$. This guarantees that $x(t) > x(1) = 2$ when $t > 1$, for the positivity of the derivative on this interval means that $x(t)$ is an increasing function there. On the other hand, $\frac{dx}{dt} = \tan(e^{-t}) \le \tan(1) < 1.56$ when $0 \le t \le 1$. Hence $x(1) - x(t) = \int_t^1 \tan(e^{-s})\,ds \le \int_t^1 \tan(1)\,dx < 1.56\,(1 - t)$ whenever $0 \le t \le 1$. It follows that if $0 \le t \le 1$ we must have $2 - (1.56 - 1.56\,t) < x(t)$, or $0 < 0.44 + 1.56\,t < x(t)$. It follows that $x(t) = 0$ is also impossible for $0 \le t \le 1$.**

Problem 3:

- **a)**

- **Answer: If $y = a\,x^2$, then $x = 0$ gives both $y = 0$ and $y' = 0$, so that condiion (i) is satisfied. However, $y' = 2\,a\,x$, and if $x = 4$ this gives $1 = y' = 8\,a$, by condition (ii), so that $a = 1/8$. But then, using the other part of condition (ii), we find that $1 = y = \frac{1}{8}\,(4)^2 = 2$, which is not possible. The curve $y = a\,x^2$ therefore can't satisfy conditions (i) and (ii) with the same choice of a.**

- **b)**

- **Answer: Let $g(x) = c\,x^3 - \frac{x^2}{16}$. Then condition (i) requires that $1 = g(4) = 64\,c - 1$, so that $c = \frac{1}{32}$. Then $g'(x) = \frac{3}{32}\,x^2 - \frac{x}{8}$, and $g'(4) = \frac{3}{32} \cdot 16 - \frac{4}{8} = \frac{3}{2} - \frac{1}{2} = 1$. Thus, g satisfies condition (ii) if we take $c = \frac{1}{32}$.**

- **c)**

- **Answer: We have $g(x) = \frac{x^3}{32} - \frac{x^2}{16} = \frac{1}{16}\left(\frac{x^3}{2} - x^2\right)$. Thus, $g'(x) = \frac{1}{16}\left(\frac{3}{2}\,x^2 - 2\,x\right) = \frac{x^2}{16}\left(\frac{3}{2}\,x - 1\right)$. Consequently, $g'(x) < 0$ for $0 < x < \frac{3}{2}$, and this means that g cannot be increasing over the interval from $x = 0$ to $x = 4$.**

- **d)**

- **Let $h(x) = x^n / k$, where k is a nonzero constant and n is a positive integer. If h meets condition (ii), then $1 = h(4) = 4^n / k$, while $1 = h'(x) = n\,4^{n-1} / k$ as well. Thus, $k = 4^n$ and $k = n\,4^{n-1}$. Thus $1 = k / k = 4^n / (n\,4^{n-1})$, and $n = 4$. But if $n = 4$ and $1 = 4^4 / k$, then $k = 4^4 = 256$. Thus, condition (ii) means that $h(x) = x^4 / 256$. That $h(0) = 0 = h'(0)$ is immediate, and this is condition (i). Also, $h'(x) = x^3 / 64 > 0$ whenever $0 < x$, and this means h is increasing betweeen $x = 0$ and $x = 4$. So condition (iii) is met.**

Problem 4:

a)

The point $(22,\ f(22))$ lies at the midpoint of the segment connecting $(20, 15)$ and $(24, 3)$, which has slope $(15 - 3)/(20 - 24) = -3$. The segment is part of the tangent line to f at $(22,\ f(22))$.

- **Answer: $f'(22) = -3$ calories per minute per minute.**

b)

- **Answer: f is increasing only on the intervals $[0, 4]$ and $[12, 16]$. On the latter interval, its rate of increase is $(15 - 9)/(16 - 12) = 3/2$ calories per minute per minute. When $0 \le x \le 4$, we have $f'(x) = -\frac{3}{4}t^2 + 3t$; $f''(t) = -\frac{3}{2}t + 3$. Now $f''(t) = 0$ when $t = 2$, $f''(t) > 0$ when $0 < t < 2$, and $f''(t) < 0$ when $2 < t < 4$. Thus f' is an increasing function on the interval $[0, 2]$, and is a decreasing function onf the interval $[2, 4]$. It follows that f' has a maximum for the interval $[0, 4]$ when $t = 2$, and this maximum value is $f'(2) = 3$. This is larger than $f'(t)$ when $12 < t < 16$, so the maximal value of $f'(t)$, i.e., the maximal rate of increase for the rate at which calories are burned is 3 calories per minute per minute at time $t = 2$.**

c)

The total number of calories burned over the time interval $6 \le t \le 18$ is $\int_6^{18} f(t)\,dt$. Computing the areas of the relevant rectangles and the relevant trapezoid, we find that this is $6 \cdot 9 + 4 \cdot (15 + 9)/2 + 2 \cdot 15 = 132$ calories.

- **Answer: 132 calories.**

d)

We must have $\frac{1}{18-6}\int_6^{18}[f(t) + c]\,dt = 15$. But $\frac{1}{12}\int_6^{18}[f(t) + c]\,dt = \frac{1}{12}\int_6^{18} f(t)\,dt + \frac{1}{12}\int_6^{18} c\,dt = \frac{132}{12} + c = 11 + c$. Thus, $11 + c = 15$, and $c = 4$.

■ **Answer:** $c = 4$.

Problem 5:

■ a)

If $f'(x) = 2 f(x)(3 - x)$, then $f'(x)/f(x) = 2(3 - x)$. Thus $\int_4^x f'(s)/f(s)\,ds = 2\int_4^x (3 - s)\,ds$. Performing the integrations, we find that $\ln|f(x)| - \ln|f(4)| = -(3 - x)^2 + (3 - 4)^2$, or $\ln|f(x)| = 1 - (3 - x)^2 = -x^2 + 6x - 8$. Consequently, $|f(x)| = e^{-x^2+6x-8}$. The right-hand side of this latter equation is always positive, and we are given that $f(4) = 1$. Hence $f(x) > 0$ for x near 4, and we may write $f(x) = e^{-x^2+6x-8}$.

■ **Answer:** $f(x) = e^{-x^2+6x-8}$.

■ b)

■ **Answer:** If $g(4) = 1$, then $\lim_{x\to\infty} g(x) = 3$ and $\lim_{x\to\infty} g'(x) = 0$.

■ c)

If $y' = 2y[3 - y]$, then $y'' = 2y'[3 - y] - 2y\,y' = 6y' - 4y\,y' = 2y'[3 - 2y]$. We know that the zeros of $y' = 2y[3 - y]$ are the asymptotes of solutions, and are not eligible for consideration as ordinates of inflection points. Consequently, the inflection points of solutions are to be found at those values of y for which $3 - 2y$ changes sign, or when $y = 3/2$.

■ **Answer:** $y = 3/2$; when $y = 3/2$, $y' = 2y(3 - y) = 2 \cdot \frac{3}{2}\left(3 - \frac{3}{2}\right) = \frac{9}{2}$.

Problem 6:

■ a)

We may find the Maclaurin series for $f'(x)$ by differentiating that for $f(x)$ term by term. Hence, the Maclaurin series for $f'(x)$ is $-3x^2 + 6x^5 - 9x^8 + \cdots + (-1)^n 3n x^{3n-1} + \cdots$.

- **Answer:** **The Maclaurin series for $f'(x)$ is $-3x^2 + 6x^5 - 9x^8 + \cdots + (-1)^n 3n x^{3n-1} + \cdots$.**

b)

The interior of the interval of convergence for the series of Part a) of this problem is identical with the interior of the interval where the Macluarin series for $f(x)$ converges. Hence, the above series converges to $f'(x)$ for all $x \in (-1, 1)$, and, in particular, for $x = \frac{1}{2}$. But $f'(x) = -\frac{3x^2}{(1+x^3)^2}$, so that $-\frac{3}{2^2} + \frac{6}{2^5} - \frac{9}{2^8} + \cdots + (-1)^n \frac{3n}{2^{3n-1}} + \cdots = f'(\frac{1}{2}) = -(3/4)/(9/8)^2 = -\frac{16}{27}$.

- **Answer:** $\sum_{k=1}^{\infty} (-1)^k 3k \left(\frac{1}{2}\right)^{3k-1} = -\frac{16}{27}$

c)

We may find the series for $\int_0^x f(t)\,dt$ by integrating the series for f term by term. Thus, the required series is $x - \frac{x^4}{4} + \frac{x^7}{7} - \frac{x^{10}}{10} + \cdots + (-1)^n \frac{x^{3n+1}}{3n+1} + \cdots$.

- **Answer:** **The Maclaurin series for $\int_0^x f(t)\,dt$ is $x - \frac{x^4}{4} + \frac{x^7}{7} - \frac{x^{10}}{10} + \cdots + (-1)^n \frac{x^{3n+1}}{3n+1} + \cdots$.**

d)

- **Answer:** **We know that the Maclaurin series for $\int_0^x f(t)\,dt$ represents the function in the interval $(-1, 1)$. Hence we may write $\int_0^x f(t)\,dt \sim x - \frac{x^4}{4} + \frac{x^7}{7}$. When $x = \frac{1}{2}$, the series for $\int_0^x f(t)\,dt$ is an alternating series, and the magnitude of difference between the sum of the first three terms and the integral is at most the magnitude of the fourth term, which is $\frac{1}{10}\left(\frac{1}{2}\right)^{10} = \frac{1}{10240} < \frac{1}{10000}$. Consequently the value of $\int_0^{1/2} f(t)\,dt$ is within $\frac{1}{10000}$ of $\frac{1}{2} - \frac{1}{2^4 \cdot 4} + \frac{1}{2^7 \cdot 7} = \frac{435}{896}$.**