Solutions to the 2007 AP Calculus BC Exam Free Response Questions

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Part A

Problem 1.

a)

The limits of the integral that gives the area are the solutions of the equation $2 = \frac{20}{1+x^2}$:

In[1]:= $\texttt{Solve}\left[2 == \frac{20}{1 + x^2}, x\right]$

Out[1]= `{{x → -3}, {x → 3}}`

The required area is therefore

In[2]:= $\int_{-3}^{3} \left(\frac{20}{1 + x^2} - 2\right) dx$

Out[2]= `-12 + 40 ArcTan[3]`

The required area is $40 \arctan 3 - 12$.

b)

Using the method of washers, we find that the required volume is

In[3]:= $\pi \int_{-3}^{3} \left(\left(\frac{20}{1 + x^2}\right)^2 - 2^2\right) dx$

Out[3]= `π (96 + 400 ArcTan[3])`

The required volume is $\pi(96 + 400 \arctan 3)$.

c)

The diameter of the semicircle at $x = t$ is $\frac{20}{1+t^2} - 2$, so the radius is $\frac{10}{1+t^2} - 1$. Hence the area $A(t)$ of the cross section at $x = t$ is $A(t) = \frac{\pi}{2}\left(\frac{10}{1+t^2} - 1\right)^2$. The required volume is therefore

In[4]:= $$\frac{\pi}{2}\int_{-3}^{3}\left(\frac{10}{1+t^2} - 1\right)^2 dt$$

Out[4]= $$\frac{1}{2}\pi\,(36 + 60\,\texttt{ArcTan}[3])$$

The required volume is $\frac{\pi}{2}(36 + 60 \arctan 3)$.

Problem 2

a)

The amount of water that enters the tank during the time interval $0 \le t \le 7$ is $\int_0^7 f(t)\,dt = \int_0^7 \left[100\,t^2 \sin\sqrt{t}\,\right] dt$:

In[5]:= $$\texttt{NIntegrate}\left[100\,t^2\,\texttt{Sin}\left[\sqrt{t}\right],\ \{t,\ 0,\ 7\}\right]$$

Out[5]= 8263.80654266

That's about 8263.806 gallons, or, to the nearest gallon, 8264 gallons.

b)

From the graph and what we are given about the intersection points of the curves, we see that the rate at which water leaves the tank exceeds that at which it leaves the tank on the intervals $[0,\ 1.617)$ and $(3,\ 5.076)$. It follows that the amount of water in the tank is decreasing on each of the intervals $[0,\ 1.617]$ and $[3,\ 5.076]$.

c)

The rate at which the amount of water in the tank increases is, as we have seen in part b), negative on the interval $(3,\ 5.076)$. It is positive on the interval $(1.617,\ 3)$. By the First Derivative Test, the amount of water in the tank has a local maximum at $t = 3$. We must also consider the amount of water in the tank when $t = 0$ and the amount when $t = 7$. Over the interval $[0,\ 7]$, $3 \times 250 + 4 \times 2000 = 8750$ gallons of water have left the tank, while, according to part a), 8263.806 gallons have entered. When $t = 7$, the amount of water in the tank is therefore $5000 + 8263.806 - 8750 = 4513.806$ gallons. The amount of water in the tank at $t = 3$ is $5000 + \int_0^{3.000}\left[100\,t^2 \sin\sqrt{t} - 250\right] dt$:

In[6]:= $$5000 + \texttt{NIntegrate}\left[100\,t^2\,\texttt{Sin}\left[\sqrt{t}\right] - 250,\ \{t,\ 0,\ 3\}\right]$$

Out[6]= 5126.59080072

The maximum occurs when $t = 3$ and, to the nearest gallon, is 5127 gallons.

Problem 3

a)

The required area is $\frac{1}{2}\int_{-2\pi/3}^{2\pi/3} r^2\, d\theta + \frac{1}{2}\int_{2\pi/2}^{4\pi/3} r^2\, d\theta = 2\int_{-2\pi/3}^{2\pi/3} d\theta + \frac{1}{2}\int_{2\pi/3}^{4\pi/3} (3 + 2\cos\theta)^2 d\theta$.

In[7]:= $$2\int_{-2\pi/3}^{2\pi/3} d\theta + \frac{1}{2}\int_{2\pi/3}^{4\pi/3} (3 + 2\,\mathrm{Cos}[\theta])^2\, d\theta$$

Out[7]= $$\frac{8\pi}{3} + \frac{1}{2}\left(-11\sqrt{3} + \frac{22\pi}{3}\right)$$

In[8]:= `% // Together`

Out[8]= $$\frac{1}{6}\left(-33\sqrt{3} + 38\pi\right)$$

The area is $\left(38\pi - 33\sqrt{3}\right)/6$.

b)

In[9]:= `r[θ_] = 3 + 2 Cos[θ]`

Out[9]= `3 + 2 Cos[θ]`

In[10]:= `r'[θ]`

Out[10]= `-2 Sin[θ]`

In[11]:= `r'[π / 3]`

Out[11]= $-\sqrt{3}$

When $\theta = \pi/3$, $dr/d\theta = -\sqrt{3}$, so at that instant, $dr/dt = dr/d\theta = -\sqrt{3}$. This means that the particle is tracing out the curve in such a way that when it moves through the point whose polar coordinates are $(4, \pi/3)$, the radial component of its velocity vector points toward the origin and has magnitude $-\sqrt{3}$.

c)

We have $y(\theta) = r(\theta)\sin\theta$:

In[12]:= `y[θ_] = r[θ] Sin[θ]`

Out[12]= `(3 + 2 Cos[θ]) Sin[θ]`

In[13]:= `y'[θ]`

Out[13]= $\mathrm{Cos}[\theta]\,(3 + 2\,\mathrm{Cos}[\theta]) - 2\,\mathrm{Sin}[\theta]^2$

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In[14]:= y'[π/3]

         1
Out[14]= ─
         2
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At the instant when $\theta = \pi/3$, $dy/dt = dy/d\theta = 1/2$. This means that the vertical component of the velocity vector points upward and has magnitude $1/2$.

Part B

Problem 4

a)

If $f(e) = 2$ and $f'(x) = x^2 \ln x$, then $f'(e) = e^2$. The eqution of the tangent line to the graph of f at the point $(e, 2)$ is therefore $y = 2 + e^2(x - e)$.

b)

Because $f'(x) = x^2 \ln x$, we have $f''(x) = x(1 + 2\ln x)$. When $x > 1$, $\ln x > 0$, so $f''(x) > 0$ on $(1, 3)$. Thus, the curve $y = f(x)$ has a graph that is concave upward on $(1, 3)$.

c)

We let $u = \ln x$, $dv = x^2\, dx$. Then $du = dx/x$ and $v = x^3/3$. Thus, integrating by parts, $\int x^2 \ln x\, dx = \frac{x^3}{3} \ln x - \int \frac{x^2}{3}\, dx = \frac{x^3}{3} \ln x - \frac{x^3}{9} + c$. But we are given that $f(e) = 2$, so $2 = \frac{e^3}{3} - \frac{e^3}{9} + c$, or $c = 2 - \frac{2}{9} e^3$. Thus, $f(x) = \frac{x^3}{3} \ln x - \frac{x^3}{9} + 2 - \frac{2}{9} e^3$.

Problem 5

a)

The linearization of r at $t = 5$ is the linear function $L(t) = r(5) + r'(5)(t - 5) = 30 + 2(t - 5)$. An approximate value for $r(5.4)$ is $L(5.4) = 30 + 2(5.4 - 5) = 30.8$ feet. The curve $r = r(t)$ is given to be concave downward, so the tangent line at each point lies (locally) above the curve. Thus, our estimate of 30.8 ft is an overestimate.

b)

Because $V(t) = \frac{4}{3} \pi [r(t)]^3$, we have $V'(t) = 4\pi [r(t)]^2 r'(t)$. Thus, $V'(5) = 4\pi\, 30^2 \cdot 2.0 \sim 22,619.467\ \text{ft}^3/\text{min}$.

c)

The right Riemann sum corresponding to the data given is $4.0\,(2-0) + 2.0\,(5-2) + 1.2\,(7-5) + 0.6\,(11-7) + 0.5\,(12-11) = 19.3$ feet. By the Fundamental Theorem of Calculus, $\int_0^{12} r'(t)\,dt = r(12) - r(0)$ is the change, in feet, in the radius from its value when $t = 0$ to its value when $t = 12$.

d)

The function r is given concave down, so r' is a decreasing function. Consequently, $r'(t) \geq r'(b)$ when t lies anywhere in an interval $[a, b]$. It follows that each term of the right Riemann sum is less than the area under the curve in the corresponding interval. The right Riemann sum therefore underestimates the integral.

Problem 6

a)

In general, $e^u = 1 + u + u^2/2! + u^3/3! + \ldots$. Thus, $e^{-x^2} = 1 - x^2 + x^4/2! - x^6/3! + \ldots$.

b)

We have $\lim_{x\to 0} \frac{1-x^2-f(x)}{x^4} = \lim_{x\to 0} \frac{1-x^2-\left(1-x^2+x^4/2-x^6/6+\ldots\right)}{x^4}$.

But the latter reduces to $\lim_{x\to 0}\left(-1/2 + x^2/6 - \ldots\right) = -1/2$.

c)

Integrating the series from Part a) term by term, we obtain a series expression for the integral: $\int_0^x e^{-t^2}\,dt = x - x^3/3 + x^5/10 - x^7/42 + \ldots$. Substituting $x = 1/2$ into the first two terms of this series, we obtain the approximation $\int_0^{1/2} e^{-t^2}\,dt \sim \frac{1}{2} - \frac{1}{24} = \frac{11}{24}$.

d)

The n-th term of the series for $\int_0^x e^{-t^2}\,dt$ is $(-1)^{n-1}\,\frac{x^{2n-1}}{(2n-1)\cdot(n-1)!}$. When $x = 1/2$, these terms decrease to zero as $n \to \infty$. It follows from the Alternating Series Test that the error resulting from truncating the series at term 2 does not exceed the magnitude of term 3, which is $1/320 < 1/200$.