

# Solutions to the 2007 AP Calculus BC (Form B) Exam Free Response Questions

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Part A

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## Problem 1

### ■ a)

The curve  $y = e^{2x-x^2}$  intersects the line  $y = 2$  where  $2x - x^2 = \ln 2$ :

```
In[1]:= Solve[2 x - x^2 == Log[2], x]
```

```
Out[1]:= {{x -> 1 - Sqrt[1 - Log[2]]}, {x -> 1 + Sqrt[1 - Log[2]]}}
```

Thus, the area of the region  $R$  is  $\int_a^b (e^{2x-x^2} - 2) dx$ , where  $a = 1 - \sqrt{1 - \ln 2}$  and  $b = 1 + \sqrt{1 - \ln 2}$ . The integral must be done numerically:

```
In[2]:= NIntegrate[e^{2x-x^2} - 2, {x, 1 - Sqrt[1 - Log[2]}, 1 + Sqrt[1 - Log[2]]}]
```

```
Out[2]:= 0.51414278561
```

The area of region  $R$  is, to three digits to the right of the decimal, **0.514**.

### ■ b)

The curve  $y = e^{2x-x^2}$  intersects the line  $y = 1$  where  $2x - x^2 = \ln 1 = 0$ , or at  $x = 0$  and at  $x = 2$ . Thus, the sum of the areas of the regions  $R$  and  $S$  is the integral  $\int_0^2 (e^{2x-x^2} - 1) dx$ . From this we subtract the integral of Part a) to obtain the area of the region  $S$ . Once again, we must integrate numerically:

```
In[3]:= NIntegrate[e^{2x-x^2} - 1, {x, 0, 2}] -
      NIntegrate[e^{2x-x^2} - 2, {x, 1 - \sqrt{1 - \text{Log}[2]}, 1 + \sqrt{1 - \text{Log}[2]} }]
```

```
Out[3]= 1.54601415295
```

The area of region  $S$  is, to three digits to the right of the decimal, **1.546**.

### ■ c)

Using the method of washers, the volume of the solid generated when the region  $R$  is rotated about the line  $y = 1$  is

$$\pi \int_a^b \left[ (e^{2x-x^2} - 1)^2 - 1 \right] dx, \text{ where } a = 1 - \sqrt{1 - \ln 2}, \text{ and } b = 1 + \sqrt{1 - \ln 2}.$$

Evaluation of this integral is not required, but let's do it numerically anyway:

```
In[4]:= \pi NIntegrate[(e^{2x-x^2} - 1)^2 - 1, {x, 1 - \sqrt{1 - \text{Log}[2]}, 1 + \sqrt{1 - \text{Log}[2]} }]
```

```
Out[4]= 4.1466068494
```

## Problem 2

### ■ a)

For an object moving along a curve in the  $x$   $y$ -plane with position given by  $(x(t), y(t))$ , speed is given by  $\sqrt{[x'(t)]^2 + [y'(t)]^2}$ :

```
In[5]:= \sqrt{\left(\text{ArcTan}\left[\frac{t}{1+t}\right]\right)^2 + (\text{Log}[t^2 + 1])^2} /. t -> 4
```

```
Out[5]= \sqrt{\text{ArcTan}\left[\frac{4}{5}\right]^2 + \text{Log}[17]^2}
```

### ■ b)

Total distance traveled is the integral of speed with respect to time. We integrate numerically:

```
In[6]:= NIntegrate[\sqrt{\left(\text{ArcTan}\left[\frac{t}{1+t}\right]\right)^2 + (\text{Log}[t^2 + 1])^2}, {t, 0, 4}]
```

```
Out[6]= 6.42330415534
```

Correct to three digits to the right of the decimal, total distance traveled is **6.423**.

### ■ c)

To find  $x(4)$ , we integrate (numerically, again)  $x'(t)$  and add to the initial value given for  $x(0)$ :

```
In[7]:= -3 + NIntegrate[ArcTan[ $\frac{t}{1+t}$ ], {t, 0, 4}]
```

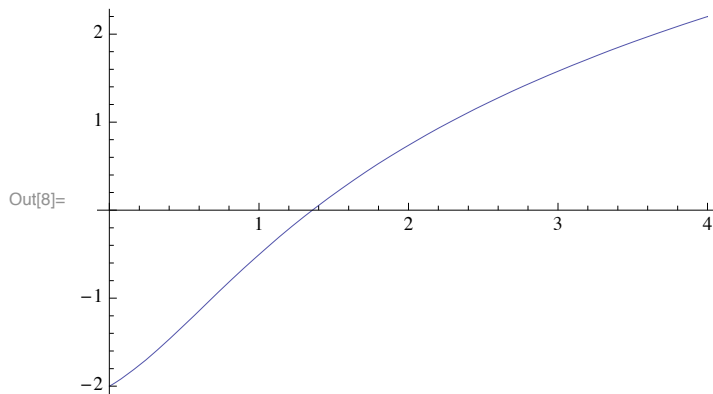
```
Out[7]= -0.89205877667
```

Correct to three digits to the right of the decimal,  $x(4) = -0.892$ .

#### ■ d)

The slope of the tangent line to the curve is given by  $(dy/dt)/(dx/dt)$ , so this ratio must be 2 at the point we are to find. We solve numerically, using a plot to suggest an initial guess of  $t = 1.4$ :

```
In[8]:= Plot[Log[t2 + 1] / ArcTan[ $\frac{t}{1+t}$ ] - 2, {t, 0, 4}]
```



```
In[9]:= FindRoot[Log[t2 + 1] / ArcTan[ $\frac{t}{1+t}$ ] == 2, {t, 1.4}]
```

```
Out[9]= {t -> 1.35766313878}
```

The object is at the point where the tangent line has slope 2 when, correct to three digits to the right of the decimal,  $t = 1.358$ .

The acceleration vector at this point is  $\langle x''(t), y''(t) \rangle$ , with  $t$  set to this value.

```
In[10]:= {D[ArcTan[ $\frac{t}{1+t}$ ], t], D[Log[t2 + 1], t]} /. %
```

```
Out[10]= {0.13510182205, 0.955008192419}
```

The acceleration vector at the point where the slope of the tangent line is 2 is  $\langle 0.135, 0.955 \rangle$ , correct to three digits to the right of the decimal.

## Problem 3

#### ■ a)

We are given  $W(v) = 55.6 - 22.1 v^{0.16}$ .

```
In[11]:= W[v_] = 55.6 - 22.1 v0.16
```

```
Out[11]= 55.6 - 22.1 v0.16
```

In[12]:=  $W'[20]$

Out[12]:=  $-0.285526919601$

$W'(20) = -0.285527$  deg/mi/hr, so that for small changes in wind velocity, the approximate change in wind chill is  $-0.286$  degrees Fahrenheit for each mile per hour change in wind velocity.

### ■ b)

Average rate of change of  $W$  over the interval from  $v = 5$  to  $v = 60$  is  $[W(60) - W(5)]/[60 - 5]$ :

In[13]:=  $\frac{W[60] - W[5]}{60 - 5}$

Out[13]:=  $-0.253796734102$

The required average rate of change is, correct to three digits to the right of the decimal,  $-0.254$ . We solve the equation  $W'(v) = [W(60) - W(5)]/[60 - 5]$  numerically to find the value of  $v$  at which instantaneous rate of change is the same as the average rate of change:

In[14]:=  $\text{NSolve}\left[W'[v] == \frac{W[60] - W[5]}{60 - 5}, v\right]$

Out[14]:=  $\{\{v \rightarrow 23.0110260815\}\}$

The wind velocity at which the instantaneous rate of change is equal to the average rate of change is  $v = 23.011$ . Note: Although we have given three digits to the right of the decimal, the precision of the numbers given in the statement of the problem makes this level of precision to be of doubtful merit in the answer.

### ■ c)

At time  $t = 0$ , wind velocity is given as 20 mph, and we are also given that it increases at a constant rate of 5 mph per hour. Thus,  $v(t) = 20 + 5t$ . We obtain  $dW[v(t)]/dt$  and evaluate it when  $t = 3$ :

In[15]:=  $v[t_] = 20 + 5t$

Out[15]:=  $20 + 5t$

In[16]:=  $D[W[v[t]], t]$

Out[16]:=  $-\frac{17.68}{(20 + 5t)^{0.84}}$

In[17]:=  $D[W[v[t]], t] /. t \rightarrow 3$

Out[17]:=  $-0.892205925875$

When  $t = 3$ , the rate of change of wind chill with respect to time is  $-0.892$  deg F/hr.

## Part B

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**Problem 4**
**■ a)**

Local maxima are to be found at points where the derivative undergoes a sign change from positive to negative. For this function, that happens **when  $x = -3$  and when  $x = 4$ .**

**■ b)**

Any point where the derivative changes from increasing to decreasing, or vice versa, is an inflection point. For this function, we find such points at  $x = -4$ ,  $x = -1$ , and  $x = 2$ .

**■ c)**

If the derivative is both positive and increasing, then the function is concave upward and has positive slope. This function displays such behavior on the intervals  $(-5, -4)$  and  $(1, 2)$ . (Whether or not to include some of the end-points depends on how we define "concave upward". This varies with textbook.)

**■ d)**

The absolute minimum value of  $f(x)$  over the interval  $-5 \leq x \leq 5$  is  $f(1) = 3$ . We know that  $x = 1$  gives a local minimum because the derivative changes sign from negative to positive at  $x = 1$ . The only other possibilities for minima are at the end-points of the interval, because we have already [Part b)] identified the other critical points in the interval as the locations of relative maxima. We have  $f(-5) = 3 + \int_1^{-5} f'(x) dx = 3 - \pi/2 + 2\pi = 3 + 3\pi/2 > 3$ . (The integral is the sum of the signed areas of two semicircles.) On the other hand,  $f(5) = 3 + \int_1^5 f'(x) dx = 3 + \frac{1}{2}3 \cdot 2 - \frac{1}{2}1 \cdot 1 = 11/2 > 3$ . (This integral is the sum of the signed areas of two triangles.) It follows that **the absolute minimum value of  $x$  over the interval  $-5 \leq x \leq 5$  is  $f(1) = 3$ .**

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**Problem 5**
**■ a)**

If  $y' = 3x + 2y + 1$ , then  $y'' = 3 + 2y' = 6x + 4y + 5$ .

**■ b)**

If  $y = mx + b + e^{rx}$ , then  $y' = m + r e^{rx}$ . If this is to be a solution to the differential equation, we must have, for all  $x$ ,  $m + r e^{rx} = 3x + 2(mx + b + e^{rx}) + 1 = (3 + 2m)x + 2e^{rx} + (2b + 1)$ . Equating coefficients of  $x$ ,  $e^{rx}$ , and the constant term, we find that this can be so only if  $3 + 2m = 0$ ,  $r = 2$ , and  $2b + 1 = m$ . **Thus  $r = 2$ ,  $m = -3/2$ , and  $b = -5/4$ .** Barbara De Roes points out that  $r = 0$ ,  $m = -3/2$ ,  $b = -9/4$  also gives a solution.

c)

Taking  $f(0) = -2$ , we find that  $f'(0) = 3 \cdot 0 + 2 \cdot (-2) + 1 = -3$ . Hence,  $f(1/2) \sim -2 + (-3) \cdot (1/2) = -7/2$ . Then  $f'(1/2) \sim 3 \cdot (1/2) + 2 \cdot (-7/2) + 1 = -9/2$ , so  $f(1) \sim -7/2 + (-9/2)(1/2) = -23/4$ .

■ d)

The Euler's Method approximation to  $g(1)$  with step-size 1 must be  $g(0) + [3 \cdot 0 + 2g(0) + 1] \cdot 1 = k + [0 + 2k + 1]$ . If this is to be 0, we must have  $3k + 1 = 0$ , or  $k = -1/3$ .

## Problem 6

■ a)

The Taylor expansion for  $e^u$  about  $u = 0$ , is  $1 + u + u^2/2! + u^3/3! + \dots$ , with general term  $u^n/n!$ . Consequently, the expansion for  $6e^{-x/3}$  is  $6(1 - x/3 + x^2/18 - x^3/162 + \dots) = 6 - 2x + x^2/3 - x^3/27 + \dots$ , with general term  $(-1)^n 2x^n / (3^{n-1} n!)$ ,  $n = 0, 1, 2, \dots$

■ b)

We need only integrate the terms of the preceding series, term by term, to find the required series. (We need not worry about the constant term because  $g(0) = 0$ .) The required series is therefore  $6x - x^2 + x^3/9 - x^4/108 + \dots$ , with general term  $(-1)^n 2x^{n+1} / [3^{n-1} (n+1)!]$ ,  $n = 0, 1, 2, \dots$

■ c)

If  $h(x) = 1 + x + x^2/2! + x^3/3! + \dots$ , then  $h(x) = e^x$ . If  $f(x) = 6e^{-x/3}$ , then  $f'(x) = -2e^{-x/3}$ . Thus,  $kf'(ax) = -2ke^{-ax/3}$ , so the equation  $h(x) = kf'(ax)$  becomes  $e^x = -2ke^{-ax/3}$ . This is possible for all  $x$  only if  $-2k = 1$  and  $-a/3 = 1$ . Thus,  $k = -1/2$  and  $a = -3$ .