Solutions to the 2008 AP Calculus BC Exam Free Response Questions

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Part A

Problem 1.

a)

The area of the given region R is $\int_0^2 \left[\sin(\pi x) - \left(x^3 - 4x\right)\right] dx$.

$$\int_0^2 \left(\text{Sin}[\pi x] - \left(x^3 - 4x\right)\right) dx$$

4

b)

We must first solve the equation $x^3 - 4x = -2$ to find the limits of integration:

$$\text{Solve}\left[x^3 - 4x == -2,\ x\right]$$

$$\left\{\left\{x \to \frac{\left(-9 + i\sqrt{111}\right)^{1/3}}{3^{2/3}} + \frac{4}{\left(3\left(-9 + i\sqrt{111}\right)\right)^{1/3}}\right\},\right.$$

$$\left\{x \to -\frac{\left(1 + i\sqrt{3}\right)\left(-9 + i\sqrt{111}\right)^{1/3}}{2\ 3^{2/3}} - \frac{2\left(1 - i\sqrt{3}\right)}{\left(3\left(-9 + i\sqrt{111}\right)\right)^{1/3}}\right\},$$

$$\left.\left\{x \to -\frac{\left(1 - i\sqrt{3}\right)\left(-9 + i\sqrt{111}\right)^{1/3}}{2\ 3^{2/3}} - \frac{2\left(1 + i\sqrt{3}\right)}{\left(3\left(-9 + i\sqrt{111}\right)\right)^{1/3}}\right\}\right\}$$

N[%]

$$\left\{\left\{x \to 1.6751309 - 1.110223 \times 10^{-16}\, i\right\}, \{x \to 0.53918887 + 0.\, i\}, \left\{x \to -2.2143197 - 5.5511151 \times 10^{-17}\, i\right\}\right\}$$

Although the solutions appear to be complex, the imaginary parts are very tiny and result from numerical errors. We need the roots that lie in the interval [0, 2], so we will ignore the third solution.

The area of that part of the region R which lies below the horizontal line $y = -2$ is given by $\int_{0.539}^{1.675} \left[-2 - \left(x^3 - 4x\right)\right] dx$.

▪ c)

The area of a cross-section of the solid perpendicular to the x-axis at $x = t$ is $\left[\sin(\pi t) - \left(t^3 - 4t\right)\right]^2$. Thus, the volue of the solid is $\int_0^2 \left[\sin(\pi t) - \left(x^3 - 4t\right)\right]^2 dt$:

$$\int_0^2 \left(\mathrm{Sin}[\pi\, t] - \left(t^3 - 4\,t\right)\right)^2 dt$$

$$\frac{1129}{105} - \frac{24}{\pi^3}$$

Numerically, this is

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N[%]
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9.9783441
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▪ d)

Under the conditions given, the pool is a region in 3-space whose base is R and whose cross-section perpendicular to the x-axis at $x = t$ has area $\left[\sin(\pi t) - \left(t^3 - 4t\right)(3 - t)\right.$. The volume is thus $\int_0^2 \left[\sin(\pi t) - \left(t^3 - 4t\right)[(3 - t)\, dt\right.$:

$$\int_0^2 \left(\mathrm{Sin}[\pi\, t] - \left(t^3 - 4\,t\right)\right)(3 - t)\, dt$$

$$\frac{116}{15} + \frac{2}{\pi}$$

Numerically,

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N[%]
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8.3699531
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Problem 2.

▪ a)

At 5:30pm, the rate at which the number of people standing in line was changing was approximately $[L(7) - L(4)]/(7 - 4) = (150 - 126)/(7 - 4) =$ 8 people per hour.

■ b)

The average number of people standing in line during the first four hours that tickets were on sale was $\frac{1}{4-0}\int_0^4 L(t)\,dt$. From the data in the table and the trapezoid rule, that is approximately

$$\frac{1}{4-0}\left(\frac{120+156}{2}(1-0)+\frac{156+176}{2}(3-1)+\frac{176+126}{2}(4-3)\right)$$

$$\frac{621}{4}$$

■ c)

By the Mean Value Theorem, there must be a point $\xi_1 \in (1,\ 3)$ such that $L'(\xi_1) = \frac{L(3)-L(1)}{3-1} = \frac{176-156}{2} > 0$ and there must be a point $\xi_2 \in (3, 4)$ such that $L'(\xi_2) = \frac{L(4)-L(3)}{4-3} = \frac{126-176}{1} < 0$. Now $L''(t)$ is defined for all $t \in [0, 9]$, and this means that L' must be a continuous function on [0, 9]. By the Intermediate Value Theorem for Continuous Functions, there must be a point $\eta_1 \in (\xi_1, \xi_2)$ where $L'(\eta_1) = 0$. By similar reasoning, there must be $\xi_3 \in (4, 7)$ where $L'(\xi_3) > 0$, and so $\eta_2 \in (\xi_2, \xi_3)$ where $L'(\eta_2) = 0$. Further, there must be $\xi_4 \in (7, 8)$ for which $L'(\xi_4) < 0$, and therefore $\eta_3 \in (\xi_3, \xi_4)$ where $L'(\eta_3) = 0$.

We conclude that $L'(t)$ takes on the value 0 at least 3 times in the interval (0, 9).

Note: It actually suffices to know that L is differentiable throughout (0, 9) to make this argument: Derivatives have the Intermediate Value Property even when they are not continuous.

■ d)

If $T(t)$ denotes the number of tickets sold at time t, we are given $T(0) = 0$ and $T'(t) = 550\,t\,e^{-t/2}$. Consequently,

$$T[t\_] = \int_0^t 550\,t\,e^{-t/2}\,dt$$

$$550\left(4 - 2\,e^{-t/2}\,(2+t)\right)$$

Thus,

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T[3.0]
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972.78412
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To the nearest whole number, 973 tickets have been sold by 3:00pm.

Problem 3.

■ a)

The first-degree Taylor polynomial for h about $x = 2$ is $T(x) = h(2) + h'(2)(x-2) = 80 + 128(x-2)$. Putting $x = 1.9$, we obtain

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T[x_] = 80 + 128 (x - 2)
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80 + 128 (−2 + x)

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T[1.9]
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67.2

Thus, our estimated value for $h(1.9)$ is 67.2. Because $h''(1) = 42$ and we are given that h'' is increasing on $1 \le x \le 3$, we know that $h''(x) > 0$ on $[1, 3]$, and from this we may infer that the graph of h is concave upward throughout this interval. Consquently, the graph of the Taylor polynomial of degree 1 at $x = 2$, which is the graph of the tangent line at $x = 2$, lies below the curve in $[1, 3]$. Consequently, our estimate of 67.2 for $h(1.9)$ is less than $h(1.9)$.

■ b)

The third degree Taylor polynomial is $T(x) = h(2) + h'(2)(x-2) + \frac{1}{2} h''(2)(x-2)^2 + \frac{1}{6} h'''(2)(x-2)^3$. Putting $x = 1.9$, we obtain

$$T[x\_] = 80 + 128 (x - 2) + \frac{1}{2}\frac{488}{3}(x-2)^2 + \frac{1}{6}\frac{448}{3}(x-2)^3$$

$$80 + 128(-2+x) + \frac{244}{3}(-2+x)^2 + \frac{224}{9}(-2+x)^3$$

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T[1.9]
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67.988444

The third degree Taylor polynomial gives an estimate of 67.988 for the value of $h(1.9)$.

■ c)

The Lagrange error estimate for the third degree Taylor polynomial $T(x)$ at $x = 2$ assures us that $\left|h(1.9) - T(1.9)\right| \le \frac{M}{24}|1.9 - 2|^4$, where M is chosen so that $\left|h^{(4)}(x)\right| \le M$ between 1.9 and 2.0. But $h^{(4)}$ is given increasing on $[1, 3]$, and we are also given tha $h^{(4)}(1) = 18$, and that $h^{(4)}(2) = \frac{584}{9}$. Consequently, $\left|h(x)\right| \le \frac{584}{9}$ when x lies between 1.9 and 2, so that we may take $M = \frac{584}{9}$. Thus, $\left|h(1.9) - T(1.9)\right| \le \frac{584}{9 \cdot 24}|1.9 - 2|^4 = \frac{584}{216}(0.1)^4 < 2.704 \times 10^{-4} < 3.000 \times 10^{-4}$.

Part B

Problem 4.

■ a)

By the Fundamental Theorem of Calculus and what we are given, we must have $x(t) = -2 + \int_0^t v(\tau)\,d\tau$. This means that $v(3) = -10$, $v(5) = -7$, and $v(6) = -9$. From the figure and the information given, we have $x'(t) = v(t) < 0$ for $0 < t < 3$ and for $5 < t < 6$, while $x'(t) > 0$ for $3 < t < 5$. Thus x is decreasing when $0 \le t \le 3$ and when $5 \le t \le 6$, while x is increasing when $3 \le t \le 5$. Thus, the particle is furthest to the left when $t = 3$ and its position at that instant is $x = -10$.

b)

Because $x(0) = -2$ and $x(3) = -10$, (see part a)), the particle moves through $x = -8$ once (leftward bound) when $0 < t < 3$. Because $x(3) = -10$, and $x(5) = -7$ (see part a)), it moves through $x = -8$ again (rightward bound) at some time in the interval $(3, 5)$. Because $x(5) = -7$ and $x(6) = -9$ (see part a) again) it moves through $x = -8$ still again (now leftward bound) at some time in the interval $(5, 6)$. The existence of these times is guaranteed, in each case, because the differentiable function x must be continuous on $[0, 6]$, and continuous functions have the intermediate value property. That these three instances are the only instances is guaranteed by the fact that x must be monotonic on each of the intervals $[0, 3]$, $[3, 5]$, and $[5, 6]$. We conclude that the particle passes through $x = -8$ just three times.

c)

Speed is the magnitude of velocity. On the interval $(2, 3)$, velocity is—from the graph—increasing, but negative. On this interval, we obtain the magnitude of velocity by reflecting the relevant portion of the curve about the t-axis. So speed is decreasing on $(2, 3)$.

d)

Acceleration is $v'(t)$. Thus, acceleration is negative on intervals where $v(t)$ is decreasing. From the graph and what we have been given about it, acceleration is negative on $[0, 1)$ and on $(4, 6]$.

Problem 5.

a)

$f'(x) = (x - 3)e^x$ is positive for $x > 3$, negative for $x < 3$, and zero when $x = 3$. Thus, f is an increasing function when $x > 3$ and a decreasing function when $x < 3$. This means that $f(3)$ is a local minimum for f.

b)

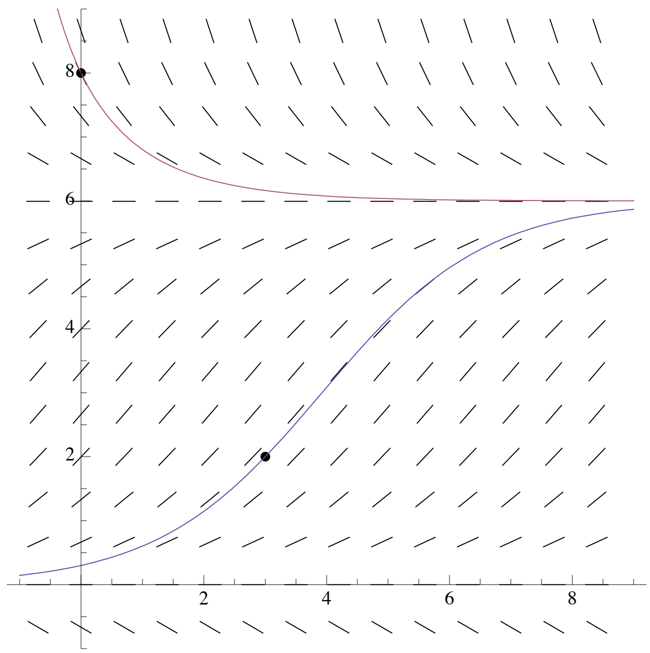
$f''(x) = (x - 2)e^x$, which is positive when $x > 2$, negative when $x < 2$. Thus, f'' is concave upward when $x > 2$ and concave downward when $x < 2$. From our analysis in part a), above, we conclude that f is both decreasing and concave upward when x is between 2 and 3.

c)

By the Fundamental Theorem of Calculus, $f(3) - f(1) = \int_1^3 f'(t)\,dt = \int_1^3 (t-3)e^t\,dt = (t-4)e^t\,\big|_1^3 = 3e - e^3$. But $f(1) = 7$ is given, so $f(3) = 7 + 3e - e^3$.

Problem 6.

a)



b)

According to Euler, if $f(t)$ is a solution of the differential equation satisfying $f(0) = 8$, then $f(1/2)$ is approximately $8 + \frac{1}{2} f'(0) = 8 + \frac{1}{2}\frac{8}{8}(6-8) = 8 - 1 = 7$. Then $f(1)$ is approximately $7 + \frac{1}{2}\frac{7}{8}(7-8) = 7 - \frac{7}{16} = \frac{105}{16}$.

c)

Because $y' = \frac{y}{8}(6-y)$, we have $y'' = \frac{1}{8} y'(6-y) - \frac{1}{8} y y' = \frac{1}{32} y(6-y)(3-y)$. Thus, $y'(0) = \frac{8}{8}(6-8) = -2$, and $y''(0) = \frac{1}{32} 8(6-8)(3-8) = \frac{5}{2}$. From this, we find that the second-degree Taylor polynomial for f about $t = 0$, is $P(t) = y(0) + y'(0) t + \frac{1}{2} y''(0) t^2 = 8 - 2t + \frac{5}{4} t^2$. Then $f(1) \simeq 8 - 2 + \frac{5}{4} = \frac{29}{4}$.

d)

This differential equation is a logistic equation with stable equilibrium solution $y(t) \equiv 6$ and unstable equilibrium solution $y(t) \equiv 0$. Thus, all positive solutions of this equation decay toward the stable solution $y(t) \equiv 6$ as $t \to \infty$. In particular, solutions of the differential equation for which $y(0) > 6$ remain larger than 6 but decrease toward $y = 6$ as $t \to \infty$, so the range of f for $t \geq 0$ is $(6, 8]$.