Solutions to the 2008 AP Calculus BC Exam Free Response Questions (Form B)

Louis A. Talman
Department of Mathematical & Computer Sciences
Metropolitan State College of Denver

Part A

Problem 1.

a)

Acceleration is the time derivative of velocity, and velocity is $\left\langle \sqrt{3\,t}\,,\, 3\cos\frac{t^2}{2} \right\rangle$. Hence, the acceleration vector is $\left\langle \frac{\sqrt{3}}{2\sqrt{t}},\, -3\,t\sin\frac{t^2}{2} \right\rangle$. When $t = 4$, acceleration is $\left\langle \frac{\sqrt{3}}{4},\, -12\sin 8 \right\rangle$.

b)

$y(0)$, the y-coordinate of position at time $t = 0$ satisfies $y(4) - y(0) = \int_0^4 y'(\tau)\,d t$. Thus, because $y(4) = 5$, $y(0)$ is given by $y(0) = 5 - 3\int_0^4 \cos\frac{t^2}{2}\,d t$. We must integrate numerically:

```
5 - NIntegrate[3 Cos[t^2/2], {t, 0, 4}]
```

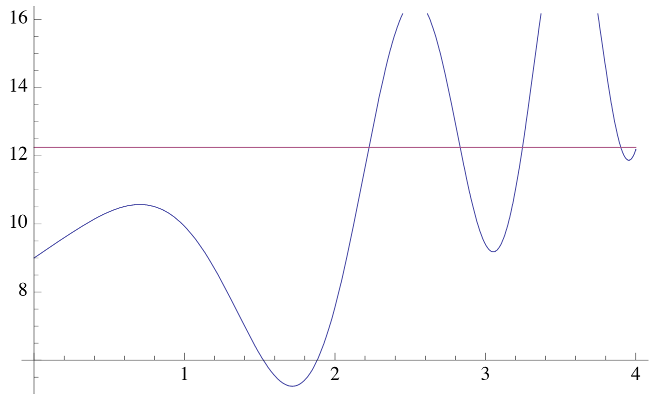
```
1.6006041
```

We conclude that the y-coordinate of position at time $t = 0$ is about 1.601.

c)

Speed is the magnitude of velocity, or $\sqrt{[x'(t)]^2 + [y'(t)]^2}$. We plot, and then solve numerically:

```
Plot[{3 t + 9 Cos[t^2/2]^2, 3.5^2}, {t, 0, 4}]
```



The first solution appears to be near $t = 2.2$.

```
FindRoot[3 t + 9 Cos[t^2/2]^2 == 3.5^2, {t, 2.2}]
{t → 2.2255812}
```

Speed first reaches 3.5 when t is approximately 2.226.

■ d)

To find total distance, we integrate speed, which is given as in part c) of this problem. The integration $\int_0^4 \sqrt{3\,t + 9\cos^2 \frac{t^2}{2}}\; dt$ must be done numerically:

```
NIntegrate[Sqrt[3 t + 9 Cos[t^2/2]^2], {t, 0, 4}]
13.18242
```

Total distance traveled over $0 \le t \le 4$ is thus about 13.182.

Problem 2

■ a)

We integrate velocity to obtain distance traveled. (The problem gives speed rather than velocity, but the given speed is never zero, and this guarantees that travel is unidirectional. We take the positive direction to be the direction of travel.) The integration $120 \int_0^2 (1 - e^{-10\,t^2})\, dt$ must be done numerically:

```
dist = NIntegrate[120 (1 - e^(-10 t^2)), {t, 0, 2}]
206.37005
```

The car travels 206.370 kilometers during the first two hours.

■ b)

We must find the value of $\frac{d}{dt} g[x(t)]$ when $t = 2$.

```
g[x_] = 0.05 x (1 - e^(-x/2))
```

$0.05\left(1-e^{-x/2}\right) x$

```
PreliminaryExpression = D[g[x[t]], t] /. t → 2
```

$0.05\left(1-e^{-\frac{x[2]}{2}}\right) x'[2]+0.025\, e^{-\frac{x[2]}{2}}\, x[2]\, x'[2]$

Now we substitute appropriate values for $x(2)$ and $x'(2)$. We obtained $x(2)$ in part a), and we are given $x'(t) = r(t)$.

```
(PreliminaryExpression /. {x[2] → dist, x'[2] → (120 (1 - e^(-10 t^2)) /. t → 2)})
```

6.

The rate of change with respect to time of the number of liters of gasoline used by the car when $t = 2$ hours is 6 liters/hour.

■ c)

We must solve $120\left(1-e^{-10 t^2}\right) = 80$ in order to find the time at which the car reaches 80 kilometers per hour:

```
Solutions = Solve[120 (1 - e^(-10 t^2)) == 80.0, t]
```

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

{{t → -0.33145321}, {t → 0.33145321}}

The error message arises because *Mathematica* is thinking about complex solutions; we want real solutions, and *Mathematica* has found all of them. We are interested only in $t > 0$, so we ignore the negative solution. We extract the positive solution:

```
b = t /. Solutions[[2]]
```

0.33145321

When we have reached a speed of 80 kilometers per hour, the distance traveled is (in kilometers) is given by an integral, as in part a):

```
dist80 = NIntegrate[120 (1 - e^(-10 t^2)), {t, 0, b}]
```

10.794097

And the amount of fuel consumed up to that time is

```
g[dist80]
```

0.53726

We've consumed 0.537 liters of fuel at the moment speed reaches 80 kilometers/hour.

Problem 3

a)

We'll store the table and define a function named **RiverDepth** taking on appropriate values:

```
T = {{0, 0}, {8, 7}, {14, 8}, {22, 2}, {24, 0}}
```

```
{{0, 0}, {8, 7}, {14, 8}, {22, 2}, {24, 0}}
```

```
Map[(RiverDepth[#[[1]]] = #[[2]]) &, T]
```

```
{0, 7, 8, 2, 0}
```

And here is the trapezoidal approximation of $\int_0^{24} \text{RiverDepth}(x)\, dx$:

```
((RiverDepth[0] + RiverDepth[8]) (8 - 0) + (RiverDepth[8] + RiverDepth[14]) (14 - 8) +
   (RiverDepth[14] + RiverDepth[22]) (22 - 14) + (RiverDepth[22] + RiverDepth[24]) (24 - 22)) / 2
```

```
115
```

The area of the cross-section is approximately 115 square feet.

b)

We integrate area times volumetric flow with respect to time and divide by the length of the time interval to obtain the average volumetric flow:

$$\frac{115}{120} \int_0^{120} \left(16 + 2\,\text{Sin}\left[\sqrt{t + 10}\right]\right) dt$$

$$\frac{23}{6} \left(480 + \sqrt{10}\ \text{Cos}\left[\sqrt{10}\right] - \sqrt{130}\ \text{Cos}\left[\sqrt{130}\right] - \text{Sin}\left[\sqrt{10}\right] + \text{Sin}\left[\sqrt{130}\right]\right)$$

Numerically, this is

```
N[%]
```

```
1807.1697
```

Average volumetric flow from $t = 0$ to $t = 120$ is 1807.170 cubic feet per minute.

c)

Once again, we integrate depth from 0 to 24:

$$\int_0^{24} 8.0\,\text{Sin}\left[\frac{\pi x}{24}\right] dx$$

```
122.231
```

Based on this model, the area of the cross-section is 122.231 square feet.

d)

We must again integrate area times volumetric flow, this time using the area given by the model, and now with t varying over the interval from 40 to 60:

$$\frac{\int_0^{24} 8.0 \,\mathrm{Sin}\left[\frac{\pi x}{24}\right] dx}{60 - 40} \int_{40}^{60} \left(16 + 2 \,\mathrm{Sin}\left[\sqrt{t + 10}\right]\right) dt$$

2181.9126

The average volumetric flow during the interval $40 \le t \le 60$ is 2181.913 cubic feet per minute. This value exceeds the given safety limit of 2100 cubic feet per minute, and indicates that water must be diverted.

Part B

Problem 4

a)

The curve $y = k x^2 - x^3 = x^2(k - x)$ crosses the x-axis at $x = 0$ and at $x = k$, and lies above the x-axis on the interval between the two when $k > 0$. Hence the area of the region R is $\int_0^k \left(k x^2 - x^3\right) dx = \frac{k}{3} k^3 - \frac{1}{4} k^4 = \frac{k^4}{12}$. This area is 2 when $k^4 = 24$, or when $k = \sqrt[4]{24}$.

b)

When the region R is rotated about the x-axis, it generates a volume given by $\pi \int_0^k \left(k x^2 - x^3\right)^2 dx$.

c)

The perimeter P of R is given by $P = k + \int_0^k \sqrt{1 + \left(2 k x - 3 x^2\right)^2}\, dx$.

Problem 5

a)

Inflection points occur at local extrema of g'. There are two such on the given graph: One at $x = 1$, and one at $x = 4$.

b)

From the picture, we see that $g'(x) < 0$ throughout the intervals $[-3, -1)$ and $(2, 6)$, while $g'(x) > 0$ throughout the intervals $(-1, 2)$ and $(6, 7]$. Consequently g is decreasing on the intervals $[-3, -1\}$ and $[2, 6]$, and is increasing on the intervals $[-1, 2]$ and $[6, 7]$. It follows that the absolute maximum value of $g(x)$ must lie either at $x = 2$ where we pass from an interval where g increases to an interval where g decreases, or at one of the endpoints of the interval $[-3, 7]$.

We are given $g(2) = 5$. Making repeated use of the fact that the area of a triangle is one-half its altitude times its base, and that the Fundamental Theorem of Calculus guarantees us that $g(x) = g(2) + \int_2^x g'(t)\,dt$, we find that $g(-3) = 5 - \left(\frac{3}{2} - 4\right) = \frac{15}{2}$ and $g(7) = g(2) + \int_2^7 g'(t)\,dt = 5 - 4 + \frac{1}{2} = \frac{3}{2}$. We now seee that $g(-3) = \frac{15}{2}$ gives the absolute maximum value for $g(x)$ when $-3 \le x \le 7$.

c)

The average rate of change of $g(x)$ on $[-3, 7]$ is $[g(7) - g(-3)]/[7 - (-3)]$. In part b), above, we found that $g(-3) = 15/2$; $g(7) = 3/2$. Thus, the average rate of change of g on $[-3, 7]$ is $(3/2 - 15/2)/(7+3) = -3/5$.

d)

The average rate of change of $g'(x)$ on $[-3, 7]$ is $[g'(7) - g'(-3)]/[7 - (-3)]$. From the graph, we have $g'(-3) = -4$; $g'(7) = 1$. Thus, the average rate of change of g' on $[-3, 7]$ is $[1 - (-4)]/(7 + 3) = 1/2$.

The Mean Value Theorem does not apply to the function g' on the interval $[-3, 7]$, because the hypotheses of that theorem require that $g''(x)$ exist for all values of $x \in (-3, 7)$. However $g''(-1)$, $g''(1)$, and $g''(4)$ do not exist for this function.

Problem 6

a)

Using the geometric series to expand $\frac{1}{1+x^2}$ in powers of x, we have $\frac{2x}{1+x^2} = 2x\left(\frac{1}{1+x^2}\right) = 2x\left(1 - x^2 + x^4 - x^6 + \ldots + (-1)^k x^{2k} + \ldots\right)$, whence the desired series expansion is $2x - 2x^3 + 2x^5 - 2x^7 + \ldots = \sum_{k=0}^{\infty} 2(-1)^k x^{2k+1}$.

b)

This series diverges when $x = 1$, because $\lim_{k\to\infty}\left[2(-1)^k 1^{2k+1}\right] \ne 0$. (In fact, the limit doesn't even exist.)

c)

We have $\int_0^x \frac{2t}{1+t^2}\,dt = \ln(1 + x^2)$. Because we have $\frac{2t}{1+t^2} = 2t - 2t^3 + 2t^5 - 2t^7 + \ldots$ (when $|t| < 1$), it follows that $\ln(1 + x^2) = \int_0^x (2t - 2t^3 + 2t^5 - 2t^7 + \ldots)\,dt = x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 - \frac{1}{4}x^8 + \ldots$ (when $|x| < 1$).

d)

From part c) of this problem, we know that $\ln\frac{5}{4} = \ln\left(1 + \left(\frac{1}{2}\right)^2\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{2}\left(\frac{1}{2}\right)^4 + \frac{1}{3}\left(\frac{1}{2}\right)^6 - \frac{1}{4}\left(\frac{1}{2}\right)^8 + \ldots$. The terms of this series are clearly decreasing in magnitude, so by the Alternating Series Test, we know that the magnitude of the error introduced by truncating the series is at most the magnitude of the first discarded term. But $\frac{1}{3}\left(\frac{1}{2}\right)^6 = \frac{1}{3\cdot 64} = \frac{1}{192} < \frac{1}{100}$, so the desired rational number A is given by $A = \left(\frac{1}{2}\right)^2 - \frac{1}{2}\left(\frac{1}{2}\right)^4 = \frac{1}{4} - \frac{1}{32} = \frac{7}{32}$.