Solutions to the 2009 AP Calculus BC Free Response Questions

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Problem 1.

■ a.

At time t = 7.5, the acceleration of Caren's bicycle is $-\frac{1}{10}$ miles per minute per minute.

■ b.

The integral $\int_0^{12} |\, v(t)\, |\; dt$ gives, in miles, the total distance that Caren traveled during the period $0 \le t \le 12$. The value of this integral is $\frac{9}{5}$ miles.

■ c.

Her turn-around time corresponds to the point on the graph where the sign of her velocity changes from positive to negative. That's $t = 2$ minutes.

■ d.

Caren lives $\int_5^{12} v(t)\, dt = \frac{7}{5}$ miles from school because she left home at $t = 5$, arrived at school at $t = 12$, traveled in one direction only, and the distance she traveled during that time is given by the integral. Larry's distance is the integral of his velocity over the interval [0, 12], or

$$\frac{\pi}{15} \int_0^{12} \text{Sin}\left[\frac{\pi}{12} t\right] dt$$

$$\frac{8}{5}$$

At $\frac{8}{5}$ miles, Larry lives farther than Caren, at $\frac{7}{5}$ miles from the school. Thus, Caren lives closer.

Problem 2.

■ a.

At time $t = 2$, the auditorium contains $\int_0^2 (1380\, t^2 - 675\, t^3)\, dt$ people.

$$\int_0^2 (1380\, t^2 - 675\, t^3)\, dt$$

980

That's 980 people.

■ b.

We are given $R(t) = 1380\, t^2 - 675\, t^3$, so that $R'(t) = 2760\, t - 2025\, t^2 = 15\, t\,(184 - 135\, t)$. Thus, $R(t)$ is increasing on the interval $[0, \frac{184}{135}]$ and decreasing on the interval $[\frac{184}{135}, 2]$, because $R'(t)$ is positive on the first of these intervals and negative on the second. It follows that the maximal rate at which people enter the auditorium is at $t = \frac{184}{135}$ hours.

■ c.

We have

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R[t_] = 1380 t^2 - 675 t^3
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$1380\, t^2 - 675\, t^3$

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Dw[t_] = (2 - t) R[t]
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$(2 - t)\,(1380\, t^2 - 675\, t^3)$

By the Fundamental Theorem of Calculus, the difference $w(2) - w(1)$ is given by

$$\int_1^2 \text{Dw}[\tau]\, d\tau$$

$$\frac{775}{2}$$

Total wait time for those who enter the auditorium after $t = 1$ is $\frac{775}{2}$ hours.

■ d.

From part (a) of this problem, above, we know that there are 980 people in the auditorium at time $t = 2$. We also know that the total wait time for these 980 people is

$$\int_0^2 \mathbf{Dw}[\tau]\, d\tau$$

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760
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760 hours. Consequently, average waiting time is $\frac{760}{980} = \frac{38}{49}$ hours.

Problem 3.

■ a.

Maximum vertical distance from the water's surface to the diver's shoulders must occur when $y'(t) = 3.6 - 9.8\,t = 0$, or when $t = \frac{18}{49}$ sec. But from $y'(t) = 3.6 - 9.8\,t$ and $y(0) = 11.4$ meters, it follows from the Fundamental Theorem of Calculus that $y(\frac{18}{49}) = 11.4 + \int_0^{18/49} (3.6 - 9.8\,t)\, dt$.

$$\mathbf{11.4} + \int_0^{18/49} \mathbf{(3.6 - 9.8\,t)}\, d\mathbf{t}$$

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12.061224
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Thus, the diver's shoulders reach a maximum distance from the water's surface of 12.061 meters.

■ b.

The diver's shoulders enter the water when $t > 0$ and $y(t) = 0$. But, as in part (a) of this problem, above,

$$\mathbf{y[t\_] = 11.4} + \int_0^t \mathbf{(3.6 - 9.8\,\tau)}\, d\tau$$

$$11.4 + 3.6\,t - 4.9\,t^2$$

$y(t) = 11.4 + \int_0^t (3.6 - 9.8\,\tau)\, d\tau = 11.4 + 3.6\,t - 4.9\,t^2$. Thus, her shoulders enter the water when $t > 0$ and $11.4 + 3.6\,t - 4.9\,t^2 = 0$

$$\mathbf{Solve[11.4 + 3.6\,t - 4.9\,t^2 == 0,\ t]}$$

$$\{\{t \to -1.2015617\}, \{t \to 1.9362556\}\}$$

That's $t = A = 1.936$ sec.

■ c.

The total distance the diver's shoulders travel from the time she leaves the platform until they enter the water is $\int_0^A \sqrt{[x'(\tau)]^2 + [y'(\tau)]^2}\, d\tau$.

```
A = t /. %[[2]]
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1.9362556
```

$$\int_0^A \sqrt{0.8^2 + (3.6 - 9.8\,\tau)^2}\, d\tau$$

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12.946211
```

Her shoulders travel 12.946 meters during the dive.

■ d.

The slope of her path at time t is $y'(t)/x'(t) = (3.6 - 9.8\,t)/0.8$, and, because this slope is negative when she enters the water, this will be the negative of the tangent of the angle which her path makes with the surface of the water at the moment of entry. Thus, the required angle is $-\text{Arctan}[y'(A)/x'(A)]$:

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-ArcTan[(3.6 - 9.8 t) / 0.8] /. t → A
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1.5188117
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At the instant her shoulders enter the water, their path makes an angle of 1.519 radians with the horizontal surface of the water.

Problem 4.

■ a.

Euler's method, with step-size h, for approximating the solution to an initial value problem $y' = g(x, y)$,
$y(a) = b$

is given by the recursion relations

$x_0 = a$;
$y_0 = b$;

$x_k = x_{k-1} + h$;
$y_k = y_{k-1} + g(x_{k-1}, y_{k-1})\, h$.

```
g[x_, y_] = 6 x^2 - x^2 y
```

$6 x^2 - x^2 y$

```
a = -1
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-1

```
b = 2
```

2

```
x[0] = a
```

-1

```
y[0] = b
```

2

```
h = 1/2
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$\frac{1}{2}$

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x[k_] := x[k - 1] + h
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y[k_] := y[k - 1] + g[x[k - 1], y[k - 1]] h
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x[1]
```

$-\frac{1}{2}$

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y[1]
```

4

```
x[2]
```

0

```
y[2]
```

$\frac{17}{4}$

Therefore, if f is the solution to this initial value problem, the Euler approximation to $f(0)$ with step-size $1/2$ is $17/4$.

▪ b.

We are given $f(-1) = 2$, $f'(-1) = 6(-1)^2 - (-1)^2 \cdot 2 = 4$, and $f''(-1) = -12$. The second degree Taylor polynomial about $x = -1$ for $f(x)$ is

$$P_2(x) = f(-1) + f'(-1)(x + 1) + \frac{1}{2!} f''(-1)(x + 1)^2 = 2 + 4(x + 1) - 6(x+1)^2.$$

[Note: If I had written this problem, this would have been part (c). Part (b) would have required the candidate to find $f''(-1)$ from the information available.]

▪ c.

From $y'(x) = 6x^2 - x^2 y(x)$, $y(-1) = 2$, we may write $\frac{y'(x)}{6 - y(x)} = x^2$. Therefore, $\int_{-1}^{x} \frac{y'(\xi)}{6 - y(\xi)} \, d\xi = \int_{-1}^{x} \xi^2 \, d\xi$, or $\int_{y(-1)}^{y(x)} \frac{d\eta}{6 - \eta} = \int_{-1}^{x} \xi^2 \, d\xi$. Equivalently, $-(\ln[6 - y(x)] - \ln[6 - 2]) = \frac{1}{3} x^3 - \frac{1}{3}(-1)^3$. Hence, $\ln[6 - y(x)] = -(\frac{1}{3} x^3 + \frac{1}{3} + \ln 4)$, so that $6 - y(x) = \exp[-(\frac{1}{3} x^3 + \ln 4 + \frac{1}{3})]$. Thus, $y(x) = 6 - 4\exp[-(\frac{1}{3} x^3 + \frac{1}{3})]$.

[Note: The Euler approximation for $y(0)$ found in part (a) misses $y(0) \sim 3.133875$ by a rather large amount. This is to be expected with a step-size so large.]

Problem 5.

▪ a.

$$f'(4) \sim \frac{f(5)-f(3)}{5-3} = \frac{-2-4}{5-3} = \frac{-6}{2} = -3.$$

▪ b.

$$\int_{2}^{13} (3 - 5 f'(x)) \, dx \quad .$$
$$= [3x - 5 f(x)] |_{2}^{13}$$
$$= [3 \cdot 13 - 5 \cdot f(13)] - [3 \cdot 2 - 5 \cdot f(2)]$$
$$= (39 - 30) - (6 - 5) = 8$$

▪ c.

The desired left Riemann sum is $f(2)(3-2) + f(3)(5-3) + f(5)(8-5) + f(8)(13-8) = 1 + 8 - 6 + 15 = 18$.

■ d.

An equation for the line tangent to the curve $y = f(x)$ at $x = 5$ is $y = f(5) + f'(5)(x - 5)$, or $y = -2 + 3(x - 5)$. Now $f''(x) < 0$ for all x in the interval $[5, 8]$, so the curve is concave downward throughout that interval; thus, the tangent line at $x = 5$ lies above the curve throughout the interval $[5, 8]$. That is, when $5 \le x \le 8$, $f(x) \le -2 + 3(x - 5)$. Consequently, $f(7) \le -2 + 3(7 - 5) = 4$.

On the other hand, $f''(x) < 0$ on $[5, 8]$ implies that the curve $y = f(x)$,being concave downward, lies above the secant line determined by the points $(5, f(5)) = (5, -2)$ and $(8, f(8)) = (8, 3)$. An equation for this secant line is $y = f(5) + \frac{f(8) - f(5)}{8 - 5}(x - 5) = -2 + \frac{5}{3}(x - 5)$. Consequently, when $5 \le x \le 8$, we have $-2 + \frac{5}{3}(x - 5) \le f(x)$. Thus, $\frac{4}{3} \le -2 + \frac{5}{3}(7 - 5) \le f(7)$.

Problem 6.

■ a.

The required Taylor expansion is

$$1 + (x - 1)^2 + \frac{1}{2!}(x - 1)^4 + \frac{1}{3!}(x - 1)^6 + \ldots + \frac{1}{n!}(x - 1)^{2n} + \ldots.$$

■ b.

The required Taylor expansion is

$$1 + \frac{1}{2!}(x - 1)^2 + \frac{1}{3!}(x - 1)^4 + \frac{1}{4!}(x - 1)^6 + \ldots + \frac{1}{(n+1)!}(x - 1)^{2n} \ldots.$$

where we have begun the series with $n = 0$, as was done in the statement of the problem.

■ c.

$$\lim_{n\to\infty} \frac{\frac{1}{(n+1)!}(x - 1)^{2n}}{\frac{1}{n!}(x - 1)^{2(n-1)}} = \lim_{n\to\infty} \frac{(x-1)^2}{n+1} = 0.$$

We conclude, by the Ratio Test, that the series converges for all values of x. That is, the interval of convergence is $(-\infty, \infty)$.

■ d.

Because the series for f converges everywhere, we may differentiate it term by term and then again, term by term. The resulting series will converge on $(-\infty,\ \infty)$ and it will represent $f''(x)$ throughout that interval. Thus, beginning with $n = 1$,

$$f''(x) = 1 + 2(x-1)^2 + \frac{5}{4}(x-1)^4 + \frac{7}{15}(x-1)^6 \ldots + \frac{2n(2n-1)}{(n+1)!}(x-1)^{2(n-1)} + \ldots .$$

But then every term of this series, except for the first, is non-negative for all x—while the first term is always positive. It follows that $f''(x) \geq 1 > 0$ for all x, and therefore that f is concave upward everywhere. f consequently has no inflection points.