Solutions to the (Form B) 2009 AP Calculus BC Free Response Questions

Louis A. Talman, Ph.D.

Department of Mathematical & Computer Sciences
Metropolitan State College of Denver

Problem 1.

a.

The area of the discarded cardboard is $600 - 20\int_0^{30} \sin(\frac{\pi x}{30})\, dx$,

$$600 - 20\int_0^{30} \mathrm{Sin}\left[\frac{\pi x}{30}\right] dx$$

$$600 - \frac{1200}{\pi}$$

which is about 218.028 square centimeters.

b.

The radius of the semicircle at $x = x_0$ is $10\sin(\frac{\pi x_0}{30})$, so the area of that semicircle is $\frac{\pi}{2}(10\sin(\frac{\pi x_0}{30}))^2$, and the volume of the cake is $\frac{\pi}{2}\int_0^{30}[10\sin(\frac{\pi x}{30})]^2\, dx$,

$$50\pi\int_0^{30}\left(\mathrm{Sin}\left[\frac{\pi x}{30}\right]\right)^2 dx$$

$$750\pi$$

or 750π cubic centimeters. At $\frac{1}{20}$ grams of unsweetened chocolate per cc, there will be $\frac{75}{2}\pi$ grams of unsweetened chocolate in the cake.

■ c.

The perimeter of the base is $30 + \int_0^{30} \sqrt{1 + \frac{4\pi^2}{9}\cos^2\left(\frac{\pi x}{30}\right)}\, dx$. This integral must be computed numerically:

$$\mathtt{30 + NIntegrate}\left[\sqrt{\mathtt{1} + \left(\frac{\mathtt{2}\,\pi}{\mathtt{3}}\,\mathtt{Cos}\left[\frac{\pi\,\mathtt{x}}{\mathtt{30}}\right]\right)^{\mathtt{2}}}\mathtt{, \{x, 0, 30\}}\right]$$

```
81.803704
```

The perimeter of the base of the cake is about 81.804 cm.

Problem 2.

■ a.

Let $D(t)$ denote the distance, in meters, from the road to the edge of the water at time t hours after the beginning of the storm. We are given $D(0) = 35$, $D'(t) = \sqrt{t} + \cos t - 3$. By the Fundamental Theorem of Calculus, $D(t) = 35 + \int_0^t \left(\sqrt{\tau} + \cos\tau - 3\right) d\tau = 35 - 3t + \frac{2}{3}t^{3/2} + \sin t$.

$$\mathtt{35 - 3\,t} + \frac{\mathtt{2}}{\mathtt{3}}\,\mathtt{t}^{\mathtt{3/2}} + \mathtt{Sin[t]\ /.\ t \to 5.0}$$

```
26.494636
```

Consequently, at the end of the 5-hour storm, we have $D(5) = 26.495$ meters.

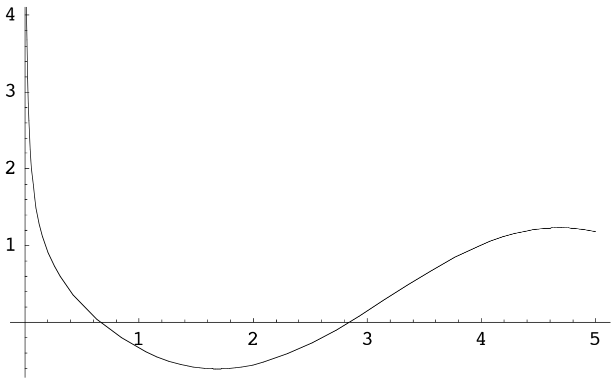
■ b.

If $f'(4) = 1.007$, then $D''(4) = 1.007$, so after four hours of the storm, the rate at which distance from road to water is changing is increasing at 1.007 meters per hour per hour.

■ c.

We are to find the absolute minimum of $f(t)$ on the interval [0, 5]. Such a minimum lies at a critical point or an end-point. The critical points for f are the zeros of $f'(t) = \frac{1}{2\sqrt{t}} - \sin t$.

$$\texttt{Plot}\left[\frac{1}{2\sqrt{\texttt{t}}} - \texttt{Sin[t], \{t, 0, 5\}}\right]$$



```
- Graphics -
```

$$\texttt{FindRoot}\left[\frac{1}{2\sqrt{\texttt{t}}} - \texttt{Sin[t], \{t, 1.0\}}\right]$$

```
{t → 0.66186548}
```

$$\texttt{FindRoot}\left[\frac{1}{2\sqrt{\texttt{t}}} - \texttt{Sin[t], \{t, 3.0\}}\right]$$

```
{t → 2.8403832}
```

$$\texttt{f[t\_] = }\sqrt{\texttt{t}}\texttt{ + Cos[t] - 3}$$

$$\texttt{-3 + }\sqrt{\texttt{t}}\texttt{ + Cos[t]}$$

The values of f at the end-points and critical points are

```
Map[f, {0, 0.66186548, 2.8403832, 5.0}]
```

```
{-2, -1.3976017, -2.2696347, -0.48026984}
```

The smallest of these is -2.2696347, so the distance from water to road was decreasing most rapidly about 2.840 hours after the storm began.

▪ d.

If sand is restored to the beach in such a way that the rate of change of the distance from water to road is $g(p)$ meters per day, where p is the number of days since pumping began, then, by the Fundamental Theorem of Calculus, P, the number of days of pumping required to restore the original distance between road and water, satisfies (approximately) the equation $35 = 26.495 + \int_0^P g(p)\,dp$. For an exact equation, we can replace 26.495 with $35 + \int_0^5 \left(\sqrt{\tau} + \cos\tau - 3\right) d\tau$.

Problem 3.

We don't appear to have been given quite enough information to solve this problem. We must assume that the line segment and the curved portion of the curve meet at the point (0, 2). In what follows, we will make this assumption.

▪ a.

The line segment that gives the portion of the curve that lies to the left of the y-axis has slope very close to $\frac{2}{3}$, so $\lim_{h\to 0^-} \frac{f(h)-f(0)}{h} \sim \frac{2}{3}$, while it is apparent from the graph that $\lim_{h\to 0^+} \frac{f(h)-f(0)}{h} < 0$. Consequently the left-hand and right-hand limits of the difference quotient do not agree; $f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h}$ cannot exist, and f is not differentiable at $x = 0$.

▪ b.

The average rate of change of f over the interval $[a, 6]$ is $\frac{f(a)-f(6\}}{a-6}$. This can be zero only if $f(a) = f(6) = 1$ while $a \neq 6$. The horizontal line through $(6, f(a)) = (6, 1)$ intersects the curve in just two other points, so there are just two values of a for which the average rate of change of f over $[a, 6]$ is zero.

▪ c.

We note that f is continuous on $[3, 6]$ and differentiable on $(3, 6)$, so the Mean Value guarabntees that there is a number $c \in [3, 6]$ such that $f'(c) = \frac{f(6)-f(3)}{6-3} = \frac{1}{3}$. Thus, we may take $a = 3$.

▪ d.

If $g(x) = \int_0^x f(t)\,dt$, then $g'(x) = f(x)$, and $g''(x) = f'(x)$. Thus g is concave upward on intervals where $f'(x) > 0$, or on intervals where f is increasing. We conclude that g is concave upward on $(-4, 0)$ and on $(3, 6)$. Whether or not we may conclude that g is concave upward on $[-4, 0]$ or on $[3, 6]$ depends upon which of several definitions of concavity we choose.

Problem 4.

▪ a.

The arc that forms the lower boundary of the region S is traced out when $0 \leq \theta \leq \frac{\pi}{3}$. (To see why this is so, simply note that $r = 1 - 2\cos\theta < 0$ precisely when $0 < \theta < \frac{\pi}{3}$.) Consequently, the required area is $\frac{1}{2}\int_0^{\pi/3} (1 - 2\cos\theta)^2\,d\theta$.

Evaluation of the integral isn't required. However, for the curious, it's $\frac{1}{2}\left(\pi - \frac{3\sqrt{3}}{2}\right)$, or about 0.272.

■ b.

We have $x = r\cos\theta = (1 - 2\cos\theta)\cos\theta$, and $y = r\sin\theta = (1 - 2\cos\theta)\sin\theta$. Therefore, $\frac{\text{dx}}{\text{d}\theta} = -\sin\theta + 4\cos\theta\sin\theta$ and $\frac{\text{dy}}{\text{d}\theta} = \cos\theta + 2\sin^2\theta - 2\cos^2\theta$.

■ c.

We have $\frac{\text{dy}}{\text{dx}} = \left(\frac{\text{dy}}{\text{d}\theta}\right)/\left(\frac{\text{dx}}{\text{d}\theta}\right) = (\cos\theta + 2\sin^2\theta - 2\cos^2\theta)/(4\sin\theta\cos\theta - \sin\theta)$. When $\theta = \pi/2$, this reduces to $2/(-1) = -2$. Also, when $\theta = \pi/2$, we have $x = (1 - 2\cos\frac{\pi}{2})\cos\frac{\pi}{2} = 0$, and $y = (1 - 2\cos\frac{\pi}{2})\sin\frac{\pi}{2} = 1$. The required equation is therefore $y = 1 - 2(x - 0)$, or $2x + y = 1$.

Problem 5.

■ a.

If $g(x) = \exp[f(x)]$, then $g'(x) = f'(x)\exp[f(x)]$, so $g'(1) = f'(1)\exp[f(1)] = -4e^2$. Hence, an equation for the line tangent to $y = g(x)$ at $x = 1$ is $y = g(1) + g'(1)(x - 1)$, or $y = e^2 - 4e^2(x - 1)$.

■ b.

By the First Derivative Test, g has a local maximum at any point where $g'(x)$ changes sign from positive to negative. But $g'(x) = f'(x)\exp[f(x)]$, and $\exp[f(x)]$ is always positive. Therefore the local maxima of g are to be found at points where $f'(x)$ changes sign from positive to negative. From the graph given, we see that there is just one such point: $x = -1$. The function g therefore has a local maximum only at $x = -1$ in the interval $(-1.2, 3.2)$.

■ c.

Because $g''(x) = \exp[f(x)]\,[(f'(x))^2 + f''(x)]$ and $\exp[f(x)] > 0$, the sign of $g''(x)$ is the same as the sign of $[(f'(x))^2 + f''(x)]$. Now, as is given, $(f'(-1))^2 = 0$, and we see from the graph that f' being a decreasing function in a neighborhood of $x = -1$ it must be the case that $f''(-1) < 0$. Consequently, $g''(-1) < 0$.

■ d.

The average rate of change of g' over the inverval $[1, 3]$ is $\frac{g'(3) - g'(1)}{3 - 1}$. Now, $g'(1) = -4e^2$, as we saw in part (a) of this problem, above. We also have $g'(3) = f'(3)\exp[f(3)] = 0$. ($f'(3) = 0$ is given.) The desired average rate of change is therefore $\frac{0 - (-4e^2)}{2} = 2e^2$.

Problem 6.

▪ a.

The given series is a geometric series with common ratio $x + 1$; it therefore converges when $|x + 1| < 1$, or when $-2 < x < 0$.

▪ b.

$1 + r + r^2 + r^3 + \ldots = \frac{1}{1 - r}$ when $|r| < 1$, so $f(x) = 1 + (x + 1) + (x + 1)^2 + \ldots = \frac{1}{1 - (x + 1)} = -\frac{1}{x}$ when $-2 < x < 0$.

▪ c.

Because $f(x) = -\frac{1}{x}$ when $-2 < x < 0$, we may write $g(-\frac{1}{2}) = \int_{-1}^{-1/2} f(t)\,dt = -\int_{-1}^{-1/2} \frac{dt}{t} = -\ln|-\frac{1}{2}| + \ln|-1| = \ln 2$.

▪ d.

When $-2 < x^2 - 1 < 0$, or, equivalently, when $-1 < x < 1$, we may replace the x in the given series with $x^2 - 1$. Noting that $(x + 1)$ becomese $[(x^2 - 1) + 1] = x^2$ under this substitution, we find that $h(x) = f(x^2 - 1) = 1 + x^2 + x^4 + x^6 + \ldots + x^{2n} + \ldots = \sum_{n=0}^{\infty} x^{2n}$. Because $x = 1/2$ lies in the interval $(-1, 1)$, we may write $h(1/2) = f\left[\left(\frac{1}{2}\right)^2 - 1\right] = f(-\frac{3}{4}) = -\frac{1}{-(3/4)} = \frac{4}{3}$.

Alternately, we may write $h(x) = f(x^2 - 1) = \frac{1}{1 - x^2}$, and, appealing once again to our knowledge of the geometric series, conclude that $h(x) = 1 + x^2 + x^4 + x^6 + \ldots + x^{2n} + \ldots$ as long as $-1 < x < 1$.