

# Lecture XI: Related Rates

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## 1 Examples

### 1.1 Problem:

*A ship left port at noon, and steamed east at seven knots. A second ship left the same port at 2 pm and steamed north at nine knots. How fast was the distance between the ships changing at 5 pm?*

In order to understand this problem better, we draw a picture and label the changing quantities with convenient names. It is very important that the picture depict the *general* situation, and not the specific situation at 5 pm. We let  $x$  denote the distance from port to the eastbound ship,  $y$  the distance from port to the northbound ship, and  $D$  the distance between the two ships—all at the same arbitrary time  $t$  (Figure 1). Of course, we see now

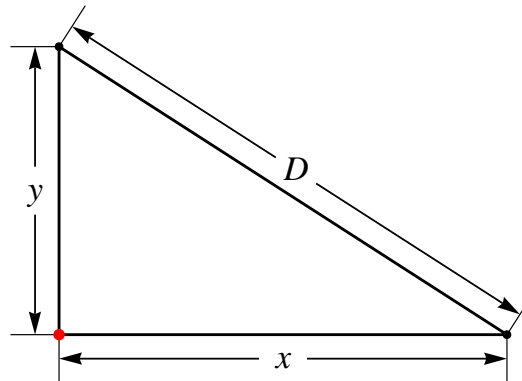


Figure 1: The two ships in general position

by the Pythagorean theorem that

$$D^2 = x^2 + y^2, \tag{1}$$

and thus, by implicit differentiation, that

$$2D \frac{dD}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}, \tag{2}$$

or

$$D \frac{dD}{dt} = x \frac{dx}{dt} + y \frac{dy}{dt}. \tag{3}$$

Now that we've obtained a relationship involving all of the important derivatives, we're ready to think about the critical instant. At five o'clock, the eastbound ship is 35 nautical miles from port, so that  $x = 35$ . At the same time, the northbound ship is 27 nautical miles from port, or  $y = 27$ . Moreover, we are given  $dx/dt = 7$  and  $dy/dt = 9$ . From equation (1), we have

$$D = \sqrt{35^2 + 27^2} \tag{4}$$

$$= \sqrt{1954} \sim 44.2 \text{ nautical miles} \tag{5}$$

at five o'clock. We substitute all of this information into (3) to obtain

$$\sqrt{1954} \frac{dD}{dt} = 35 \cdot 7 + 27 \cdot 9 = 488, \tag{6}$$

from whence

$$\frac{dD}{dt} = \frac{488}{\sqrt{1954}} \sim 11.04 \text{ knots}. \tag{7}$$

At five o'clock, the distance between the two ships is increasing at a rate of just over 11 knots.

## 1.2 Problem:

*A particle is moving counterclockwise around the circle of radius 2 centered at the origin. Its speed is constant and it makes a complete revolution every 4 seconds. How quickly is the distance from the particle to the point (4,0) changing when the particle is moving through the point with coordinate (0,2)?*

A diagram helps to clarify our thinking. See Figure 2.

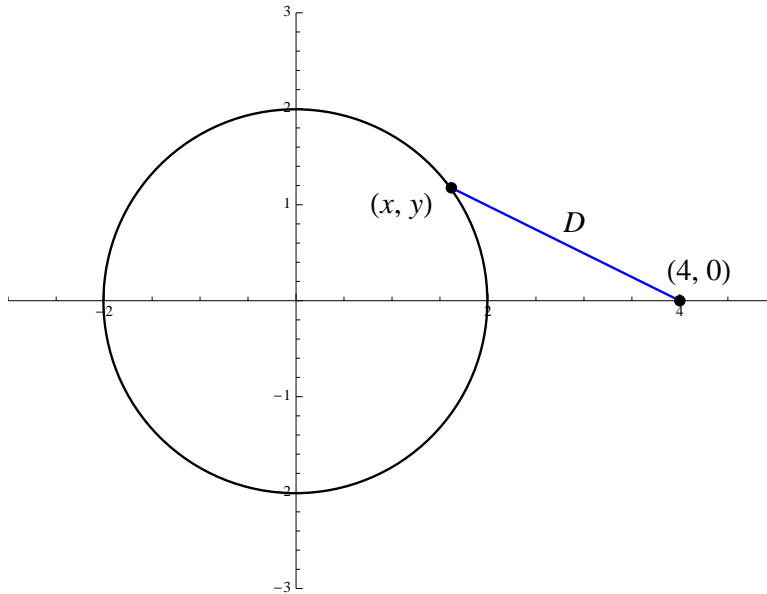


Figure 2: A particle moving in a circular path

By the distance formula,

$$D^2 = (x - 4)^2 + y^2. \quad (8)$$

Here,  $x$ ,  $y$  and  $D$  are the varying quantities that interest us. They are all functions of time, related by equation (8). If we differentiate this equation implicitly, we obtain a relationship involving the derivatives of these functions:

$$2D \frac{dD}{dt} = 2(x - 4) \frac{dx}{dt} + 2y \frac{dy}{dt}, \quad (9)$$

or

$$D \frac{dD}{dt} = (x - 4) \frac{dx}{dt} + y \frac{dy}{dt}. \quad (10)$$

At the critical instant, we know that  $x = 0$  and  $y = 2$ . Thus, at that instant,  $D = \sqrt{(0 - 4)^2 + (2 - 0)^2} = 2\sqrt{5}$ . The particle has a constant speed that carries it around the circle in just four seconds, so that speed is  $\pi$  units per second. At the critical instant, as the particle crosses the  $y$ -axis, it is moving directly to the left and neither up nor down. Hence,  $dx/dt = -4$  and  $dy/dt = 0$ . Substituting these values for  $x$ ,  $y$ ,  $D$ ,  $dx/dt$ , and  $dy/dt$  into equation (10), we find that

$$2\sqrt{5} \cdot \frac{dD}{dt} = (0 - 4) \cdot (-4) + 2 \cdot 0, \quad (11)$$

whence

$$\frac{dD}{dt} = \frac{8}{\sqrt{5}} = \frac{8\sqrt{5}}{5}. \quad (12)$$

The distance from  $(4, 0)$  to the particle is changing at the rate of  $8/\sqrt{5}$  units per second. That's approximately 3.58 units per second.

### 1.3 Problem

A 10-foot ladder rests against a wall. The base of the ladder is sliding away from the wall at  $1/2$  foot/second. A rung seven feet from the base of the ladder is painted red. How quickly is the distance from the red rung to the floor changing when the base of the ladder is 6 feet from the wall?

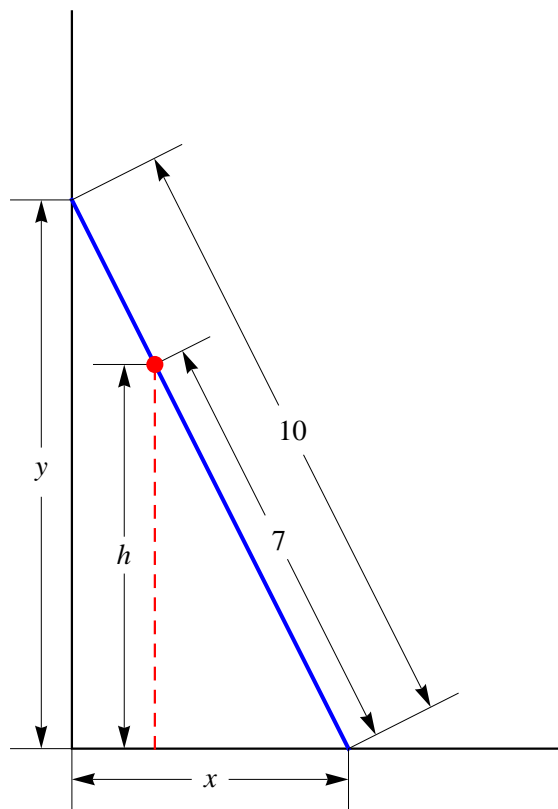


Figure 3: The 10-foot ladder in general position; all distances in feet

Once again, we begin with a diagram. (See Figure 3.) From the picture we've drawn of the ladder in general position, we can now write relationships between various different changing quantities. Because our picture depicts *general* position, we can write relationships that are true all the time, and not just at one particular instant. We've been told how  $x$ , the distance from the wall to the base of the ladder, is changing, and we want to know how  $h$ , the distance from the red rung to the floor, is changing. So we need to find a relationship, or a chain of relationships, connecting  $h$  to  $x$ .

The ladder, the floor, and the wall form a right triangle, and this gives us a first relationship, via the Pythagorean theorem:

$$x^2 + y^2 = 100. \quad (13)$$

The triangle formed by the ladder, the wall, and the floor is similar to the triangle formed by the lower seven feet of the ladder, the line connecting the red rung perpendicularly to the floor, and the baseline from the foot of the perpendicular to the base of the ladder. This gives us a second relationship:

$$\frac{h}{7} = \frac{y}{10}. \quad (14)$$

Now we have two options. We can eliminate  $y$  from these equations to obtain a single equation that relates  $h$  to  $x$ , or we can use both relationships in a somewhat different way. Let's explore both options.

### 1.3.1 Option 1

By equation (13), and because  $y \geq 0$ , we may write

$$y = \sqrt{100 - x^2}. \quad (15)$$

If we substitute for  $y$  in (14) according to (15) and rearrange, we find that

$$h = \frac{7}{10} \sqrt{100 - x^2}. \quad (16)$$

Both  $h$  and  $x$  are to be understood as varying quantities—functions, in fact, of time—which are *always* related to each other by the equation (16). So we take the derivative, using implicit differentiation with respect to  $t$ , on both sides of (16). This gives us a relationship between the derivatives  $dh/dt$  and  $dx/dt$ .

$$\frac{dh}{dt} = \frac{-7x}{10\sqrt{100 - x^2}} \frac{dx}{dt}. \quad (17)$$

After we have obtained equation (17)—*and not until then*—we are ready to use the information about the values taken on by various changing quantities at the *critical instant*

when  $x = 6$ . At that instant (and, in this particular problem, at all other instants) we know that  $dx/dt = 1/2$ . Hence,

$$\left. \frac{dh}{dt} \right|_{x=6} = \frac{-7 \cdot 6}{10\sqrt{100 - 36}} \cdot \frac{1}{2} \quad (18)$$

$$= -\frac{21}{80} \text{ ft/sec.} \quad (19)$$

### 1.3.2 Option 2

Once again, we treat all of the named quantities we have introduced,  $x$ ,  $y$ , and  $h$ , as functions of time. These quantities are to be understood as *always* being related to each other through equations (13) and (14). We can obtain relations among the derivatives  $dx/dt$ ,  $dy/dt$  and  $dh/dt$  by applying implicit differentiation to both of the equations (13) and (14). Doing so leads us to

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \text{ and} \quad (20)$$

$$\frac{1}{7} \frac{dh}{dt} = \frac{1}{10} \frac{dy}{dt}. \quad (21)$$

Equation (20) tells us that

$$\frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt}, \quad (22)$$

while (21) says that

$$\frac{dh}{dt} = \frac{7}{10} \frac{dy}{dt}. \quad (23)$$

Combining these two latter equations yields

$$\frac{dh}{dt} = -\frac{7x}{10y} \frac{dx}{dt}. \quad (24)$$

At long last (and *only* at long last), we are ready for information pertaining to the *critical instant*, at which  $x = 6$  and  $dx/dt = 1/2$ . Equation (24) requires us to know the value of  $y$  at the critical instant; we can get that from (13), which tells us that  $6^2 + y^2 = 100$ , or that  $y = 8$  at the moment in question. Substituting all of this information into (24) now gives us

$$\left. \frac{dh}{dt} \right|_{x=6} = -\frac{7 \cdot 6}{10 \cdot 8} \cdot \frac{1}{2} \quad (25)$$

$$= -\frac{21}{80} \text{ ft/sec.} \quad (26)$$

### 1.3.3 Conclusion

Both methods yield the same answer (and it would be embarrassing, to say the least, if they did not). Option 1 seems the easier calculation, but we have to be careful about making too hasty a generalization. We were able to effect Option 1 in this problem because we could effect a general solution of equation (13) for  $y$  in terms of  $x$ . If that had not been the case, Option 2 would have been our only choice. *Although the technique we used in Option 1 seems the simpler, it is not always available to us.* Thus, it is important that we understand both methods of solution.

## 2 Strategy

We can't give you specific instructions for problems of this kind. The problems come in too many different forms, and the relations that connect the variable quantities are just too diverse to catalogue. But we can outline a generic procedure that should be helpful.

1. Draw a picture!! It should not depict the critical instant, but a generic instant.
2. Identify and name the changing quantities that are of interest. It is better to label too many than too few. Be generous with your labels!
3. Write one or more relationships among the variables you introduced in step 2. These relationships *must not* incorporate information from the critical instant; they must describe the *general* situation.
4. Differentiate the relationships from step 3 implicitly to obtain one or more relationships among the variables and their derivatives.
5. Substitute the information from the critical instant into the relationships you obtained in steps 3 and 4.

The third item in this list is the most important and the most difficult. We can't emphasize strongly enough: The relationships you write must describe a *general* instant, and not just the critical instant. Writing such relationships demands all of your mathematical knowledge and experience. That knowledge and experience must therefore include formulae for commonly occurring distances, areas, and volumes—among other things.