MTH 1410, Calculus I—Exam I **Feb. 18, 2009**

Instructions: Work the following problems; give your reasoning and show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Your paper is due at 2:50 pm.

1. Evaluate the following limits. Use the Limit Laws. You need not mention the Limit Laws explicitly, but you must show the calculations they lead you to.

   (a)

$$\lim_{x\to -3} \frac{x^2+x-6}{x^2+4x+3}$$

   (b)

$$\lim_{x\to\infty} \frac{(5-x)(10+8x)}{(3-3x)(3+10x)}$$

   Solution:

   (a)

$$\lim_{x\to -3} \frac{x^2+x-6}{x^2+4x+3} = \lim_{x\to -3} \frac{\cancel{(x+3)}(x-2)}{\cancel{(x+3)}(x+1)} = \frac{-5}{-2} = \frac{5}{2}.$$

   (b)

$$\begin{aligned}\lim_{x\to\infty} \frac{(5-x)(10+8x)}{(3-3x)(3+10x)} &= \lim_{x\to\infty} \frac{[(5-x)(10+8x)]/x^2}{[(3-3x)(3+10x)]/x^2} \\ &= \lim_{x\to\infty} \frac{(5/x-1)(10/x+8)}{(3/x-3)(3/x+10)} = \frac{(-1)\cdot(8)}{(-3)\cdot(10)} = \frac{4}{15}.\end{aligned}$$

2. A rock is thrown off of a 100-meter cliff with an upward velocity of 50 m/s. As a result, its height (above the ground at the base of the cliff) after t seconds is given by the formula $h(t) = 100 + 50t - 5t^2$.

   (a) What is its height after 7 seconds?

   (b) What is its average velocity over the first seven seconds?

   (c) What is its velocity after 7 seconds?

   (d) What is its velocity when it hits the ground at the base of the cliff?

   Solution:

   (a) $h(7) = 100 + 50\cdot 7 - 5\cdot 7^2 = 205$ meters.

   (b) Average velocity over the first seven seconds is

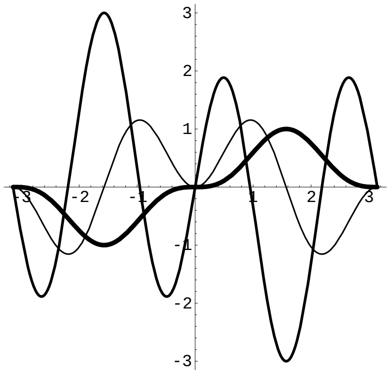
$$\frac{h(7)-h(0)}{7-0} = \frac{205-100}{7} = 15 \text{ meters per second.}$$

(c) Velocity is the derivative $h'(t) = 50 - 10t$, so when $t = 7$, velocity is $h'(7) = 50 - 10 \cdot 7 = -20$ meters per second.

(d) The rock hits the ground when $t > 0$ satisfies $h(t) = 0$. But the only positive solution of the equation $100 + 50t - 5t^2 = 0$ is $t = 5 + 3\sqrt{5}$. So the rock hits the ground at the base of the cliff with velocity

$$h'(5 + 3\sqrt{5}) = 50 - 10(5 + 3\sqrt{5}) = -30\sqrt{5} \text{ meters per second.}$$

3. Here is a graph showing three functions—a skinny one, a middle-weight one, and a fat one—on the same pair of axes:



One of the curves is f, one is f', and one is f''. Explain which is which and how you know.

Solution: The skinny curve lies above the x-axis exactly where the fat curve is increasing, touches the x-axis exactly where the fat curve has a horizontal tangent, and lies below the x-axis exactly where the fat curve is decreasing. The middle-weight curve lies above the x-axis exactly where the skinny one is increasing, touches the x-axis exactly where the skinny one has a horizontal tangent, and lies below the x-axis exactly where the skinny one is decreasing. Thus, the fat curve is f, the skinny curve is f', and the middle-weight curve is f''.

4. Find $f'(x)$ if

(a) $f(x) = 3x^2 - 4x + 5$

(b) $f(x) = \dfrac{x^2 - 2\sqrt{x}}{x^{1/3}}$

Solution:

(a) $f'(x) = 6x - 4$.

(b) Carrying out the division, we find that $f(x) = x^{2-1/3} - 2x^{1/2-1/3} = x^{5/3} - 2x^{1/6}$. Consequently, $f'(x) = \dfrac{5}{3}x^{2/3} - \dfrac{1}{3}x^{-5/6}$.

5. Find $f'(x)$ if

   (a) $f(x) = (3x^2 - x + 1)^2(5x + 4)^{12}$

   (b) $f(x) = \dfrac{a}{x^{10}} + \sin^3 bx$, where a and b are fixed but unspecified constants.

   Solution:

   (a) $f'(x) = 2(3x^2 - x + 1)(6x - 1)(5x + 4)^{12} + 60(3x^2 - x + 1)^2(5x + 4)^{11}$

   (b) $f(x) = ax^{-10} + [\sin(bx)]^3$, so $f'(x) = -10ax^{-11} + 3b\sin^2(bx)\cos(bx)$.

6. A function f is given by

$$f(x) = \begin{cases} 2cx + 2 & ; \quad x \leq 3 \\ 3 - cx & ; \quad 3 < x. \end{cases}$$

   For what values of the constant c is f continuous on $(-\infty, \infty)$? Be sure to give your reasoning.

   Solution: If $a \neq 3$ the values of the function $f(x)$ are given by a polynomial function in some open interval centered at a, so f is a continuous function everywhere except possibly at $x = 3$. In order for f to be continuous at $x = 3$, we need to have $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3) = 6c + 2$. Now $\lim_{x\to 3^-} f(x) = 6c + 2$, but $\lim_{x\to 3^+} = 3 - 3c$. Consequently, f is continuous at $x = 3$ precisely when c is chosen so that $6c + 2 = 3 - 3c$, or when $c = 1/9$.

7. (a) Use the definition of the derivative to find $f'(x)$ if $f(x) = \sqrt{2x}$.

   (b) Use the derivative you calculated in part (a) of this problem to write equations for the lines tangent to the curve $y = \sqrt{2x}$ at $x = 1$, at $x = 2$, and at $x = 8$.

   Solution:

   (a)

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{2(x+h)} - \sqrt{2x}}{h} \\ &= \lim_{h\to 0} \frac{2(x+h) - 2x}{h\left[\sqrt{2(x+h)} + \sqrt{2x}\right]} \\ &= \lim_{h\to 0} \frac{\cancel{2x} + 2\cancel{h} - \cancel{2x}}{\cancel{h}\left[\sqrt{2(x+h)} + \sqrt{2x}\right]} \\ &= \frac{2}{\sqrt{2x} + \sqrt{2x}} = \frac{1}{\sqrt{2x}}. \end{aligned}$$

   (b) i. Tangent line at $x = 1$:

$$y = f(1) + f'(1)(x - 1),$$

which is

$$y = \sqrt{2} + \frac{1}{\sqrt{2}}(x-1).$$

ii. Tangent line at $x = 2$:

$$y = f(2) + f'(2)(x-2),$$

which is

$$y = 2 + \frac{1}{2}(x-2).$$

iii. Tangent line at $x = 8$:

$$y = f(8) + f'(8)(x-8),$$

which is

$$y = 4 + \frac{1}{4}(x-8).$$

8. Let F be the function given by

$$F(x) = f(x)g(x),$$

where f and g are functions for which $f'(x)$ and $g'(x)$ are both defined for all real values of x.

(a) What is $F'(x)$?

(b) Derive the formula you have given in part **??**.

Solution:

(a) $F'(x) = f'(x)g(x) + f(x)g'(x)$.

(b)

$$\begin{aligned}
F'(x) &= \lim_{h\to0} \frac{F(x+h)-F(x)}{h} \\
&= \lim_{h\to0} \frac{f(x+h)g(x+h)-f(x)g(x)}{h} \\
&= \lim_{h\to0} \frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h} \\
&= \lim_{h\to0} \left[\frac{f(x+h)-f(x)}{h}g(x+h)\right] + \lim_{h\to0} \left[f(x)\frac{g(x+h)-g(x)}{h}\right] \\
&= \left[\lim_{h\to0} \frac{f(x+h)-f(x)}{h}\right] \lim_{h\to0} g(x+h) + \lim_{h\to0} f(x) \left[\lim_{h\to0} \frac{g(x+h)-g(x)}{h}\right],
\end{aligned}$$

provided all of the limits in the latter expression exist. But

$$\begin{aligned} \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= f'(x), \\ \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} &= g'(x), \text{ and} \\ \lim_{h\to 0} f(x) &= f(x). \end{aligned}$$

We are given that $g'(x)$ exists for all real x, and this means that the function g is continuous everywhere. Consequently,

$$\lim_{h\to 0} g(x+h) = g(x).$$

It now follows that

$$\begin{aligned} F'(x) &= \left[\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\right] \lim_{h\to 0} g(x+h) + \lim_{h\to 0} f(x) \left[\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}\right] \\ &= f'(x)g(x) + f(x)g'(x). \end{aligned}$$