MTH 1410, Calculus I—Exam II **March 18, 2009**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 2:50 pm.

1. Find $f'(x)$ if

   (a) $f(x) = 3x^6 - 14x^5 + 12x^3 + 7x^2 - 8x$.

   (b) $f(x) = \dfrac{3x^2 - 5x}{x^2 + x + 1}$.

   Solution:

   (a) $f'(x) = 18x^5 - 70x^4 + 36x^2 + 14x - 8$.

   (b) $f'(x) = \dfrac{(6x-5)(x^2+x+1) - (3x^2-5x)(2x+1)}{(x^2+x+1)^2}$.

2. Find $f'(x)$ if

   (a) $f(x) = \cos^3 x \sin 2x$.

   (b) $f(x) = \ln\left[\cos^2 2x \sin^4 x\right]$.

   Solution:

   (a) $f'(x) = -3\cos^2 x \sin x \sin 2x + 2\cos^3 x \cos 2x$.

   (b) $f'(x) = D_x(2\ln\cos 2x + 4\ln\sin x) = -\dfrac{4\sin 2x}{\cos 2x} + \dfrac{4\cos x}{\sin x} = -4\tan 2x + 4\cot x$.

3. (a) *Use the definition of the derivative* to find $f'(x)$ if $f(x) = 1/\sqrt{x}$.

   (b) Use the derivative you calculated in part (a) of this problem to write equations for the lines tangent to the curve $y = 1/\sqrt{x}$ at $x = 1$, at $x = 4$, and at $x = 9$.

   Solution:

   (a)

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}\right] \\
&= \lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x+h}\sqrt{x}} \\
&= \lim_{h\to 0} \frac{x - (x+h)}{h\sqrt{x+h}\sqrt{x}(\sqrt{x} + \sqrt{x+h})} \\
&= -\lim_{h\to 0} \frac{1}{\sqrt{x+h}\sqrt{x}(\sqrt{x} + \sqrt{x+h})} = -\frac{1}{2x\sqrt{x}}.
\end{aligned}$$

(b) $f'(1) = -1/2$, so the equation of the tangent line at $x = 1$ is $y = 1-(x-1)/2$. $f'(4) = -1/16$, so the equation of the tangent line at $x = 4$ is $y = (1/2) - (1/16)(x - 4)$. $f'(9) = -1/54$; the equation of the tangent line at $x = 9$ is $y = (1/3)-(1/54)(x-9)$.

4. Let f be the function given by

$$f(x) = \begin{cases} x^2 + 2x, & x \le 2 \\ ax^2 + b, & x > 2. \end{cases}$$

(a) What condition must the constants a and b satisfy if f is to be a continuous function?

(b) Find all pairs of values for a and b which make the function f a differentiable function.

Solution:

(a) If f is to be continuous, we must have $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x)$. But

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} ax^2 + b = 4a + b,$$

while

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^2 + 2x = 8.$$

If f is to be continuous, the constants a and b must therefore satisfy the equation $4a + b = 8$.

(b) If f is to be differentiable, f must be continuous at $x = 2$ and the derivatives from the left and from the right must match at $x = 2$. Thus, we must have

$$\begin{aligned} f'(2) &= \lim_{h\to 0^+} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h\to 0^+} \frac{(2+h)^2 + 2(2+h) - 8}{h} \\ &= \lim_{h\to 0^+} \frac{4 + 4h + h^2 + 4 + 2h - 8}{h} \\ &= \lim_{h\to 0^+} (6 + h) = 6, \end{aligned}$$

and

$$\begin{aligned} f'(2) &= \lim_{h\to 0^-} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h\to 0^-} \frac{[a(2+h)^2 + b] - (4a + b)}{h} \\ &= \lim_{h\to 0^-} (4a + ah) = 4a \end{aligned}$$

together with the equation we derived in part (a): $4a + b = 8$. Thus, $4a = 6$ and $4a + b = 8$, so that $a = 3/2$ and $b = 2$.

5. A man started walking north at 4 feet per second from a point P. Five minutes later, a woman started walking south at 5 feet per second from a point 500 feet due east of P. At what rate were the people moving apart 15 minutes after the woman started walking.

Solution: Let x denote the distance the man has walked north of the point P, and let y denote the distance the woman has walked south of her starting point. Let D be the distance between the two. By the Pythagorean Theorem, $D^2 = (x+y)^2 + 500^2 = (x+y)^2 + 250000$. We differentiate this latter equation implicitly with respect to time to obtain:

$$2D\frac{dD}{dt} = 2(x+y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right), \text{ or}$$
$$\frac{dD}{dt} = \frac{1}{D}(x+y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right).$$

We are given that $dx/dt = 4$ and $dy/dt = 5$, and this means that

$$\frac{dD}{dt} = \frac{9(x+y)}{D}.$$

At the critical instant, the woman has been walking for 15 minutes, or 900 seconds, so $y = 4500$. At that instant, the man has been walking for 20 minutes, so $x = 4800$. Thus, $D = \sqrt{(4800+4500)^2 + 250000} = 100\sqrt{8674}$, and this means that

$$\frac{dD}{dt} = \frac{9(4800+4500)}{100\sqrt{8674}} = \frac{837}{\sqrt{8674}} \text{ ft/sec.}$$

6. (a) Find an equation for the line tangent to the curve $x^3 - 4x^2y + 2xy^3 + 4 = 0$ at the point whose coordinates are $(2, 1)$.

   (b) Use the result of part (a) of this problem to find an approximate value for the y-coordinate of the point $(2.04, y)$ that lies on the curve $x^3 - 4x^2y + 2xy^3 + 4 = 0$ near the point $(2, 1)$.

Solution:

(a) Differentiating the equation $x^3 - 4x^2y + 2xy^3 + 4 = 0$ implicitly with respect to x, we find that

$$3x^2 - 8xy - 4x^2y' + 2y^3 + 6xy^2y' = 0;$$
$$y' = \frac{8xy - 3x^2 - 2y^3}{6xy^2 - 4x^2};$$
$$y'\Big|_{(2,1)} = -\frac{1}{2},$$

and the equation of the required tangent line is $y = 1 - \frac{1}{2}(x-2)$.

(b) The equation $x^3 - 4x^2y + 2xy^3 + 4 = 0$ defines y implicitly as a function φ of x near the point $(2, 1)$; we have just computed the linearization of φ at $x = 2$. When x is near 2, we may estimate $\varphi(x)$ by using the linearization in its place. Consequently,

$$\varphi(2.04) \sim 1 - \frac{1}{2}(2.04 - 2) = 1 - 0.02 = 0.98.$$