MTH 1410, Calculus I—Exam III **April 22, 2009**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 2:50 pm.

1. Find the limits:

   (a) $\displaystyle\lim_{x\to-1}\frac{x+x^2}{\ln(2+x)}$

   (b) $\displaystyle\lim_{x\to0}\frac{xe^{-2x}}{\pi e^{2x}-\pi}$

   Solution:

   (a) $\lim_{x\to-1}(x+x^2)=0=\lim_{x\to-1}\ln(2+x)$, so we may attempt l'Hôpital's rule:

   $$\begin{aligned}\lim_{x\to-1}\frac{x+x^2}{\ln(2+x)}&=\lim_{x\to-1}\frac{1+2x}{[1/(2+x)]}\\&=\lim_{x\to-1}(1+2x)(2+x)\\&=-1.\end{aligned}$$

   (b) $\lim_{x\to0}xe^{-2x}=0=\lim_{x\to0}(\pi e^{2x}-\pi)$, so we can attempt l'Hôpital's rule again:

   $$\begin{aligned}\lim_{x\to0}\frac{xe^{-2x}}{\pi e^{2x}-\pi}&=\lim_{x\to0}\frac{e^{-2x}-2xe^{-2x}}{2\pi e^{2x}}\\&=\frac{1}{2\pi}.\end{aligned}$$

2. Find the absolute maximum and the absolute minimum for the function

   $$f(x)=2x^3-3x^2-12x+20$$

   on the interval $[-3,3]$.

   Solution: Absolute extrema are to be found only at endpoints and critical numbers. We have $f'(x)=6x^2-6x-12=6(x+1)(x-2)$, which is defined everywhere and is zero only when $x=-1$ or $x=2$. Thus, the extrema are among the numbers $f(-3)$, $f(-1)$, $f(2)$, and $f(3)$. We find that $f(-3)=-25$, $f(-1)=27$, $f(2)=0$, and $f(3)=11$. The absolute minimum is $f(-3)=-25$, and the absolute maximum is $f(-1)=27$.

3. Let F be the function given by

   $$F(x)=(x-1)^2(x+1)^3.$$

   Then, in fully factored form,

   $$F'(x)=(x-1)(x+1)^2(5x-1)$$

and, also in fully factored form,

$$F''(x) = 20(x+1)\left[x - \frac{1}{5}\left(1-\sqrt{6}\right)\right]\left[x - \frac{1}{5}\left(1+\sqrt{6}\right)\right].$$

Use this information to determine the intervals where F is increasing, the intervals where F is decreasing, the intervals where F is concave upward, and the intervals where F is concave downward. What are the critical numbers of F? What is the nature of each of the critical points (local maximum, local minimum, or neither)? *Give your reasoning.*

Solution: The quantity $(x-1)$ is positive when $x > 1$ and negative when $x < 1$; $(x+1)^2$ is positive unless $x = -1$; and $(5x-1)$ is positive when $x > 1/5$, negative when $x < 1/5$. Thus, $F'(x) > 0$ when $-\infty < x < -1$, when $-1 < x < 1/5$ and when $1 < x < \infty$. It follows that F is increasing on the intervals $(-\infty, 1/5]$ and $[1, \infty)$, but decreasing on $[1/5, 1]$.

The quantity $(x+1)$ is negative when $x < -1$ and positive when $x > -1$; $[x-(1-\sqrt{6})/5]$ is negative when $x < (1-\sqrt{6})/5$ and positive when $x > (1-\sqrt{6})/5$; $[x-(1+\sqrt{6})/5]$ is negative when $x < (1+\sqrt{6})/5$ and positive when $x > (1+\sqrt{6})/5$. Consequently $F''(x) < 0$ when $-\infty < x < -1$ and when $(1-\sqrt{6})/5 < x < (1+\sqrt{6})/5$, but $F''(x) > 0$ when $-1 < x < (1-\sqrt{6})/5$ and when $(1+\sqrt{6})/5 < x < \infty$. So F is concave upward on $[-1, (1-\sqrt{6})/5]$ and on $[(1+\sqrt{6})/5, \infty)$, but concave downward on $(-\infty, -1]$ and on $[(1-\sqrt{6})/5, (1+\sqrt{6})/5]$.

From the expression for $F'(x)$, we see that the critical numbers for F are $x = 1$, $x = -1$, and $x = 1/5$. Our analysis of the increasing/decreasing behavior of F above, shows that $x = 1$ gives a local minimum for F because F is decreasing just to the left of $x = 1$ but increasing just to the right of $x = 1$. Similarly, $x = 1/5$ gives a local maximum for F, and $x = -1$ gives neither a local maximum nor a local minimum.

4. Let f be the function given by

$$f(x) = \begin{cases} x^2 + 2x, & x \le 2 \\ ax^2 + b, & x > 2. \end{cases}$$

   (a) What condition must the constants a and b satisfy if f is to be a continuous function?

   (b) Find all pairs of values for a and b which make the function f a differentiable function.

Solution:

(a) If f is to be continuous, we must have $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x)$. But

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} ax^2 + b = 4a + b,$$

while

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^2 + 2x = 8.$$

If f is to be continuous, the constants a and b must therefore satisfy the equation $4a+b = 8$.

(b) If f is to be differentiable, f must be continuous at $x = 2$ and the derivatives from the left and from the right must match at $x = 2$. Thus, we must have

$$\begin{aligned} f'(2) &= \lim_{h\to 0^+} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h\to 0^+} \frac{(2+h)^2 + 2(2+h) - 8}{h} \\ &= \lim_{h\to 0^+} \frac{4 + 4h + h^2 + 4 + 2h - 8}{h} \\ &= \lim_{h\to 0^+} (6 + h) = 6, \end{aligned}$$

and

$$\begin{aligned} f'(2) &= \lim_{h\to 0^-} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h\to 0^-} \frac{[a(2+h)^2 + b] - (4a + b)}{h} \\ &= \lim_{h\to 0^-} (4a + ah) = 4a \end{aligned}$$

together with the equation we derived in part (a): $4a + b = 8$. Thus, $4a = 6$ and $4a + b = 8$, so that $a = 3/2$ and $b = 2$.

5. Find the points on the ellipse $x^2 + 4y^2 = 4$ whose distance from the point $(1, 0)$ is minimal.

Solution: Let (x, y) be any point on the curve. Then $x^2 + 4y^2 = 4$ and the square S of the distance from (x, y) to $(1, 0)$ is $S = (x - 1)^2 + y^2$, where $|x| \leq 2$. We can minimize distance by minimizing S. Thus we want to find the critical points of S. Treating y as a function of x and differentiating, we find that $dS/dx = 2(x - 1) + 2yy'$, so want to learn where $2(x - 1) + 2yy' = 0$. From $x^2 + 4y^2 = 4$, we see that $2x + 8yy' = 0$, or $y' = -x/(4y)$. Thus, we want

$$\begin{aligned} 0 &= 2(x - 1) + 2yy' \\ &= x - 1 + y\left(-\frac{x}{4y}\right) \\ &= x - 1 - \frac{1}{4}x \\ &= \frac{3}{4}x - 1, \end{aligned}$$

so that

$$x = \frac{4}{3}.$$

The only critical number for $S(x)$ is thus at $x = 4/3$. The minimum for $S(x)$ must occur either at $x = 4/3$ or at an endpoint $x = \pm 2$. We note that from $x^2 + 4y^2 = 1$ it follows that $y = \pm\sqrt{5}/3$ when $x = 4/3$ and that $y = 0$ when $x = \pm 2$. We therefore have $S(-2) = 9$, $S(4/3) = 2/3$, and $S(2) = 1$. The minimal distance therefore occurs when $x = 4/3$ and $y = \pm\sqrt{5}/3$—that is, at the points $(4/3, \sqrt{5}/3)$ and $(4/3, -\sqrt{5}/3)$.

6. Murgatroyd was driving his car toward an intersection at 60 miles per hour. A police cruiser was approaching the same intersection but on the cross-street (which is at right angles to the road that Muratroyd is on), at 50 miles per hour. When both cars were a quarter of a mile from the intersection, a police officer in the cruiser pointed a radar gun at Murgatroyd and measured the speed at which the two cars were approaching each other. What did she get?

   Solution: Let x denote the distance from Murgatroyd to the intersection, and let y denote the distance from the police car to the interesection. The distance D between the two cars satisfies

$$D^2 = x^2 + y^2,$$

   so

$$2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt},$$

   or

$$\frac{dD}{dt} = \frac{1}{D}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right).$$

   At the critical instant, we have $x = y = 1/4$, $D = 1/(2\sqrt{2})$, $dx/dt = -60$, and $dy/dt = -50$. Thus, rate of change of the distance between the two cars was

$$\begin{aligned}\frac{dD}{dt} &= 2\sqrt{2}\left[\frac{1}{4}(-60) + \frac{1}{4}(-50)\right] \\ &= -\frac{110}{\sqrt{2}} \text{ mph.}\end{aligned}$$

   The reading on the radar gun was $110/\sqrt{2}$ mph, or about 78 mph.