MTH 1410, Calculus I—Final Exam **December 10, 2007**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 4:50 pm.

1. (a) *Use the definition of the derivative* to find $f'(x)$ if $f(x) = 1/\sqrt{x}$.

   (b) Use the derivative you calculated in part (a) of this problem to write equations for the lines tangent to the curve $y = 1/\sqrt{x}$ at $x = 1$, at $x = 4$, and at $x = 9$.

   Solution:

   (a)

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}\right] \\ &= \lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x+h}\sqrt{x}} \\ &= \lim_{h\to 0} \frac{x - (x+h)}{h\sqrt{x+h}\sqrt{x}(\sqrt{x} + \sqrt{x+h})} \\ &= -\lim_{h\to 0} \frac{1}{\sqrt{x+h}\sqrt{x}(\sqrt{x} + \sqrt{x+h})} = -\frac{1}{2x\sqrt{x}}. \end{aligned}$$

   (b) From the immediately preceding equation, $f'(1) = -1/2$, so the equation of the tangent line at $x = 1$ is $y = 1 - (x-1)/2$, or $x + 2y = 3$. $f'(4) = -1/16$, so the equation of the tangent line at $x = 4$ is $y = (1/2) - (1/16)(x-4)$, or $x + 16y = 12$. $f'(9) = -1/54$; the equation of the tangent line at $x = 9$ is $y = (1/3) - (1/54)(x-9)$, or $x + 54y = 27$.

2. Evaluate the following definite integrals. Give all of your reasoning.

   (a) $\displaystyle\int_3^5 (3x^2 - 24x + 54)\,dx$

   (b) $\displaystyle\int_0^3 3t\sqrt{9 - t^2}\,dt$

   Solution:

   (a)

$$\int_3^5 (3x^2 - 24x + 54)\,dx = (x^3 - 12x^2 + 54x)\Big|_3^5 = 14.$$

   (b) Let $u = 9 - t^2$. Then $du = -2t\,dt$, or $t\,dt = -(1/2)\,du$. Moreover, $t = 0 \Rightarrow u = 9$ and $t = 3 \Rightarrow u = 0$. Thus

$$\begin{aligned} \int_0^3 3t\sqrt{9 - t^2}\,dt &= -\frac{3}{2}\int_9^0 u^{1/2}\,du = \frac{3}{2}\int_0^9 u^{1/2}\,du \\ &= u^{3/2}\Big|_0^9 = 9^{3/2} - 0^{3/2} = 27. \end{aligned}$$

3. Let F be the function given by

$$F(x) = (x-2)^2(x+3)^3.$$

Then, in fully factored form,

$$F'(x) = 5x(x-2)(x+3)^2$$

and, also in fully factored form,

$$F''(x) = 20(x+3)\left[x - \sqrt{\frac{3}{2}}\right]\left[x + \sqrt{\frac{3}{2}}\right].$$

Use this information to determine the intervals where F is increasing, the intervals where F is decreasing, the intervals where F is concave upward, and the intervals where F is concave downward. What are the critical numbers of F? What is the nature of each of the critical points (local maximum, local minimum, or neither)? *Give your reasoning.*

Solution: The quantity x is positive when $x > 0$ and negative when $x < 0$; $(x-2)$ is positive when $x > 2$ and negative when $x < 2$; $(x+3)^2$ is positive unless $x = -3$. Thus, $F'(x) > 0$ when $-\infty < x < -3$, when $-3 < x < 0$ and when $2 < x < \infty$. It follows that F is increasing on the intervals $(-\infty, 0]$ and $[2, \infty)$, but decreasing on $[0, 2]$.

The quantity $(x+3)$ is negative when $x < -3$ and positive when $x > -3$; $\left[x - \sqrt{3/2}\right]$ is negative when $x < \sqrt{3/2}$ and positive when $x > \sqrt{3/2}$; $\left[x + \sqrt{3/2}\right]$ is negative when $x < -\sqrt{3/2}$ and positive when $x > -\sqrt{3/2}$. Consequently $F''(x) < 0$ when $-\infty < x < -3$ and when $-\sqrt{3/2} < x < \sqrt{3/2}$, but $F''(x) > 0$ when $-3 < x < -\sqrt{3/2}$ and when $\sqrt{3/2} < x < \infty$. So F is concave upward on $\left(-3, -\sqrt{3/2}\right)$ and on $\left(\sqrt{3/2}, \infty\right)$, but concave downward on $(-\infty, -1]$ and on $\left(-\sqrt{3/2}, \sqrt{3/2}\right]$.

From the expression for $F'(x)$, we see that the critical numbers for F are $x = 2$, $x = 0$, and $x = -3$. Our analysis of the increasing/decreasing behavior of F above, shows that $x = 2$ gives a local minimum for F because F is decreasing just to the left of $x = 2$ but increasing just to the right of $x = 2$. Similarly, $x = 0$ gives a local maximum for F, and $x = -3$ gives neither a local maximum nor a local minimum.

4. Suppose that $f(2) = 2$, $f(4) = 4$, $f'(2) = 4$, $f'(4) = -2$, $g(2) = 4$, $g(4) = 2$, $g'(2) = -6$, and $g'(4) = -8$.

   (a) Find $F(4)$ and $F'(4)$, where $F(x) = \dfrac{f(x)}{g(x)}$.
   (b) Find $G(2)$ and $G'(2)$, where $G(x) = g[2f(x)]$.
   (c) Find $H(2)$ and $H'(2)$, where $H(x) = g[f(x^2)]$.

Solutions:

(a)

$$F(4) = \frac{f(4)}{g(4)} = 2;$$
$$F'(4) = \frac{f'(4)g(4) - f(4)g'(4)}{[g(4)]^2} = 7.$$

(b)

$$G(2) = g[2f(2)] = 2;$$
$$G'(x) = g'[2f(x)]D_x[2f(x)] = 2g'[2f(x)]f'(x),$$

so

$$G'(2) = 2g'[2f(2)]f'(2) = -64.$$

(c)

$$H(2) = g[f(2^2)] = 2;$$
$$H'(x) = g'[f(x^2)] \cdot D_x f(x^2) = g'[f(x^2)]f'(x^2)D_x x^2 = 2xg'[f(x^2)]f'(x^2),$$

so

$$H'(2) = 4g'[f(4)]f'(4) = 64.$$

5. (a) Show that the point $(3, 2)$ lies on the curve given by the equation

$$x^3 - 5x^2y^3 + 8y^4 + 205 = 0.$$

(b) If x and y are related by the equation, $x^3 - 5x^2y^3 + 8y^4 + 205 = 0$, find the value of y' at $(3, 2)$.

(c) Show how to use the results of parts (a) and (b) of this problem to find an approximate value for y near 2 when $x = 74/25$.

Solution:

(a) When $x = 3$ and $y = 2$, we have

$$3^3 - 5 \cdot 3^2 \cdot 2^3 + 8 \cdot 2^4 + 205 = 27 - 360 + 128 + 205 = 0,$$

so the point with coordinates $(3, 2)$ lies on the curve whose equation is $x^3 - 5x^2y^3 + 8y^4 + 205 = 0$.

(b) Treating y as a function of x and differentiating implicitly gives

$$3x^2 - 10xy^3 - 15x^2y^2y' + 32y^3y' = 0;$$
$$(32y^3 - 15x^2y^2)y' = 10xy^3 - 3x^2;$$
$$y' = \frac{10xy^3 - 3x^2}{32y^3 - 15x^2y^2}.$$

Thus,

$$y'\bigg|_{(3,2)} = \frac{10 \cdot 3 \cdot 2^3 - 3 \cdot 3^2}{32 \cdot 2^3 - 15 \cdot 3^2 \cdot 2^2} = \frac{240 - 27}{256 - 540} = \frac{213}{-284} = -\frac{3}{4}.$$

(c) From the previous part of this problem, we know that the equation of the line tangent to the curve at $(3,2)$ is

$$y = 2 - \frac{3}{4}(x-3).$$

When a point (x_0, y_0) lies near $(3,2)$ on the curve $x^3 - 5x^2y^3 + 8y^4 + 205 = 0$, it lies near the line tangent to the curve at $(3,2)$. Thus, we can approximate the value of y near 2 that satisfies the equation

$$\left(\frac{74}{25}\right)^3 - 5\left(\frac{74}{25}\right)^2 y^3 + 8y^4 + 205 = 0$$

as

$$y \sim 2 - \frac{3}{4}\left(\frac{74}{25} - 3\right) = 2 + \frac{3}{4} \cdot \frac{1}{25} = \frac{203}{100}.$$

6. Find the points on the hyperbola $4y^2 - x^2 = 1$ whose distance from the point $(5,0)$ is minimal.

Solution: The squared distance D from a point (x,y) to the point $(5,0)$ is given by

$$D = (x-5)^2 + y^2.$$

The critical numbers for D are the solutions of $D' = 0$, or, by implicit differentiation, of

$$2(x-5) + 2yy' = 0. \tag{1}$$

But $4y^2 - x^2 = 1$, so by implicit differentiation again,

$$y' = \frac{x}{4y}. \tag{2}$$

Combining (2) with (1), we find that we must solve

$$(x-5) + y\left(\frac{x}{4y}\right) = 0.$$

Solving, we find that $x = 4$. To find corresponding y-values, we substitute $x = 4$ into $4y^2 - x^2 = 1$ and solve, finding that $y = \pm\sqrt{17}/2$. The points we seek are therefore the points $(4, \sqrt{17}/2)$ and $(4, -\sqrt{17}/2)$.