MTH 3440, Partial Differential Equations—Exam I **March 4, 2009**

Instructions: Work the following problems; give your reasoning and show your supporting calculations. Your paper is due at 12:50 pm.

1. Classify each of the following partial differential equations as elliptic, parabolic, or hyperbolic. *Give reasons for your choices.*

   (a) $u_{xy} = 0$.

   (b) $u_{xx} + u_{xy} + u_{yy} = 2x$.

   (c) $u_{xx} - u_{xy} + u_{yy} = 2u$.

   (d) $u_{xx} - 2u_{xy} + u_{yy} = u_y$.

   (e) $u_{xx} - u_{yy} - u_y = 0$.

   Solution: The constant-coefficients partial differential equation

   $$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

   is hyperbolic if $B^2 - 4AC > 0$, parabolic if $B^2 - 4AC = 0$, and elliptic if $B^2 - 4AC < 0$. Thus the equations above are

   (a) $B^2 - 4AC = 1 > 0$; hyperbolic.

   (b) $B^2 - 4AC = -3 < 0$; elliptic.

   (c) $B^2 - 4AC = -3 < 0$; elliptic.

   (d) $B^2 - 4AC = 0 = 0$; parabolic.

   (e) $B^2 - 4AC = 4 > 0$; hyperbolic.

2. Which of the following are solutions of the partial differential equation $u_t = \alpha^2 u_{xx}$? (Note: No boundary conditions; no initial condition.) *Give your reasoning.*

   (a) $u(x,t) = e^{-\lambda^2\alpha^2 t}(\cos\lambda x - \sin\lambda x)$, where λ is an arbitrary constant.

   (b) $u(x,t) = 3x - 2$.

   (c) $u(x,t) = 2e^{\lambda^2\alpha^2 t}\tan 2\lambda x$, where λ is an arbitrary constant.

   (d) $u(x,t) = 4\alpha^2 e^{4\alpha^2 t}\cosh 2x$.

   Solution:

   (a) This is a solution because

   $$u_t - \alpha^2 u_{xx} = -\lambda^2\alpha^2 e^{-\lambda^2\alpha^2 t}(\cos\lambda x - \sin\lambda x) - \alpha^2(-\lambda^2\cos\lambda x + \lambda^2\sin\lambda x) = 0.$$

   (b) This is a solution because both u_t and u_{xx} vanish everywhere—making $u_t = \alpha^2 u_{xx}$.

   (c) This is not a solution. $u_t = 2\alpha^2\lambda^2\tan 2\lambda x$, but $\alpha^2 u_{xx} = 16\alpha^2\lambda^2 e^{\lambda^2\alpha^2 t}\tan 2\lambda x\sec^2 2\lambda x$.

   (d) This is a solution because

   $$u_t - \alpha^2 u_{xx} = 16\alpha^4 e^{4\alpha^2 t}\cosh 2x - \alpha^2[4\alpha^2 e^{4\alpha^2 t}(4\cosh 2x)] = 0.$$

3. Suppose that $u(x,y)$ and $v(x,y)$ are indefinitely differentiable functions that meet the conditions

   $$\begin{aligned} u_x(x,y) &= v_y(x,y) \\ u_y(x,y) &= -v_x(x,y) \end{aligned}$$

everywhere in the plane. Show that $u(x,y)$ and $v(x,y)$ must both be solutions of the partial differential equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0$$

throughout the plane.

Solution: From $u_x = v_y$, we deduce that $u_{xx} = v_{yx} = v_{xy}$. But from $u_y = -v_x$, we see that $u_{yy} = -v_{xy}$. Thus, $u_{xx} = -u_{yy}$, or $u_{xx} + u_{yy} = 0$. Similarly, from $u_x = v_y$ it follows that $v_{yy} = u_{xy}$, while $u_y = -v_x$ implies that $u_{yx} = -v_{xx}$. But $u_{xy} = u_{yx}$, and thus $v_{xx} + v_{yy} = 0$.

4. Find the finite Fourier cosine transform on $[0,1]$ of the function f given by $f(x) = x - x^2$.

   Solution: When $k \neq 0$ we have

$$\begin{aligned} 2\int_0^1 (x-x^2)\cos k\pi x\,dx &= \frac{2}{k\pi}(x-x^2)\sin k\pi x\Big|_0^1 + \frac{2}{k^2\pi^2}(1-2x)\cos k\pi x\Big|_0^1 + \frac{4}{k^3\pi^3}\sin k\pi x\Big|_0^1 \\ &= \frac{2}{k^2\pi^2}[(-1)^{k+1}-1] \\ &= \begin{cases} -4/(k^2\pi^2), & k \text{ even, } k \neq 0, \\ \\ 0, & k \text{ odd.} \end{cases} \end{aligned}$$

   When $k = 0$, $2\int_0^1 (x-x^2)\cos k\pi x\,dx = 2\int_0^1 (x-x^2)\,dx = 1/3$.

5. Find the positive eigenvalues and the eigenfunctions of the Sturm-Liouville problem

$$\begin{aligned} X''(x) + \lambda X(x) &= 0; \\ X(0) &= 0; \\ X'(1) &= 0. \end{aligned}$$

   Solution: Solutions of the differential equation corresponding have the form $X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$. From the boundary conditions, we find that we must have

$$\begin{aligned} &0 = X(0) = A, \text{ and} \\ &0 = X'(1) = \sqrt{\lambda}B\cos\sqrt{\lambda}. \end{aligned}$$

   It follows that $\sqrt{\lambda_n} = (2n-1)\pi/2$ for positive integers n, or that $\lambda_n = (2n-1)^2\pi^2/4$. These are the eigenvalues; the corresponding eigenfunctions are given by $X_n(x) = \sin\sqrt{\lambda_n}x$.

6. Show how to transform the IBVP

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, && 0 < x < 1,\ 0 < t < \infty, \\ u(0,t) &= \cos\pi t, && 0 < t < \infty, \\ u(1,t) &= \sin\pi t, && 0 < t < \infty, \\ u(x,0) &= x, && 0 < x < 1, \end{aligned}$$

   into an IBVP with homogeneous boundary conditions. (*Do not* attempt to solve the transformed problem.)

   Solution: This is Problem 2 of Problem Set 2. See the solutions for that problem set.