MTH 3440, Partial Differential Equations—Exam II **April 22, 2009**

Instructions: Work the following problems; give your reasoning and show your supporting calculations. Your paper is due at 12:50 pm.

1. Classify each of the following partial differential equations as elliptic, parabolic, or hyperbolic. *Give reasons for your choices.*

   (a) $u_{xy} = 0$.

   (b) $u_{xx} + u_{xy} + u_{yy} = 2x$.

   (c) $u_{xx} - u_{xy} + u_{yy} = 2u$.

   (d) $u_{xx} - 2u_{xy} + u_{yy} = u_y$.

   (e) $u_{xx} - u_{yy} - u_y = 0$.

2. Show that $u(x,t) = \cos x \sin t$ is a solution to the problem

$$\begin{aligned} u_{tt} &= u_{xx}, & 0 < x, 0 < t \\ u(x,0) &= 0, & 0 < x \\ u_t(x,0) &= \cos x, & 0 < x \\ u_x(0,t) &= 0, & 0 < t. \end{aligned}$$

3. Solve:

$$\begin{aligned} u_{tt} &= 4u_{xx}, & -\infty < x < \infty, 0 < t \\ u(x,0) &= 2\sin x \cos x, & -\infty < x < \infty \\ u_t(x,0) &= \cos x, & -\infty < x < \infty. \end{aligned}$$

4. Which of the following are solutions of $\nabla^2 u = 0$ on the open unit disk? (Be sure to give your reasoning.)

   (a) $u(x,y) = x^5 - 10x^3y^2 + 5xy^4$

   (b) $u(x,y) = \cosh x \sin y + \cos x \sinh y$

   (c) $u(r,\theta) = r^2 \cos 2\theta$

   (d) $u(r,\theta) = \ln r$

5. Show how to use separation of variables to find the Fourier expansion of the solution for the interior Dirichlet problem on the polar unit disk $\{(r,\theta)\colon 0 \le r < 1\}$:

$$\begin{aligned} \nabla^2 u &= 0, & 0 \le r < 1 \\ u(1,\theta) &= g(\theta), \end{aligned}$$

   where g is continuous and periodic of period 2π. You may assume that the general solution to the Euler equation $r^2R'' + rR' - \lambda^2 R = 0$ is $R = a + b\ln r$ when $\lambda = 0$ and $R = ar^\lambda + br^{-\lambda}$ when $\lambda \neq 0$. It may also be helpful to recall that, in polar coordinates,

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$