MTH 3440, Partial Differential Equations—Exam II **April 22, 2009**

Instructions: Work the following problems; give your reasoning and show your supporting calculations. Your paper is due at 12:50 pm.

1. Classify each of the following partial differential equations as elliptic, parabolic, or hyperbolic. *Give reasons for your choices.*

   (a) $u_{xy} = 0$.

   (b) $u_{xx} + u_{xy} + u_{yy} = 2x$.

   (c) $u_{xx} - u_{xy} + u_{yy} = 2u$.

   (d) $u_{xx} - 2u_{xy} + u_{yy} = u_y$.

   (e) $u_{xx} - u_{yy} - u_y = 0$.

   Solution: The constant-coefficients partial differential equation

   $$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

   is hyperbolic if $B^2 - 4AC > 0$, parabolic if $B^2 - 4AC = 0$, and elliptic if $B^2 - 4AC < 0$. Thus the equations above are

   (a) $B^2 - 4AC = 1 > 0$; hyperbolic.

   (b) $B^2 - 4AC = -3 < 0$; elliptic.

   (c) $B^2 - 4AC = -3 < 0$; elliptic.

   (d) $B^2 - 4AC = 0 = 0$; parabolic.

   (e) $B^2 - 4AC = 4 > 0$; hyperbolic.

2. Show that $u(x,t) = \cos x \sin t$ is a solution to the problem

   $$\begin{aligned} u_{tt} &= u_{xx}, & 0 < x, 0 < t \\ u(x,0) &= 0, & 0 < x \\ u_t(x,0) &= \cos x, & 0 < x \\ u_x(0,t) &= 0, & 0 < t. \end{aligned}$$

   Solution: If $u(x,t) = \cos x \sin t$, then

   $$\begin{aligned} u_x(x,t) &= -\sin x \sin t, \\ u_{xx}(x,t) &= -\cos x \sin t, \\ u_t(x,t) &= \cos x \cos t, \\ u_{tt}(x,t) &= -\cos x \sin t. \end{aligned}$$

   Thus,

   $$\begin{aligned} u_{xx}(x,t) &= -\cos x \sin t = u_{tt}(x,t), \\ u(x,0) &= -\sin x \sin 0 = 0, \\ u_t(x,0) &= \cos x \cos 0 = \cos x, \text{ and} \\ u_x(0,t) &= -\sin 0 \sin t = 0. \end{aligned}$$

3. Solve:

$$\begin{aligned} u_{tt} &= 4u_{xx}, & -\infty < x < \infty, 0 < t \\ u(x,0) &= 2\sin x \cos x, & -\infty < x < \infty \\ u_t(x,0) &= \cos x, & -\infty < x < \infty. \end{aligned}$$

Solution: Solutions to the wave equation $u_{tt} = cu_{xx}$ have the form

$$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi,$$

where f and g are the initial data for u and u_t respectively. The solution to the given problem is therefore

$$\begin{aligned} u(x,t) &= \frac{1}{2}[\sin 2(x-2t) + \sin 2(x+2t)] + \frac{1}{4}\int_{x-2t}^{x+2t} \cos\xi\, d\xi \\ &= \frac{1}{2}[\sin 2(x-2t) + \sin 2(x+2t)] + \frac{1}{4}\sin\xi \Big|_{x-2t}^{x+2t} \\ &= \frac{1}{2}[\sin 2(x-2t) + \sin 2(x+2t)] + \frac{1}{4}[\sin(x+2t) - \sin(x-2t)]. \end{aligned}$$

4. Which of the following are solutions of $\nabla^2 u = 0$ on the open unit disk? (Be sure to give your reasoning.)
   (a) $u(x,y) = x^5 - 10x^3y^2 + 5xy^4$
   (b) $u(x,y) = \cosh x \sin y + \cos x \sinh y$
   (c) $u(r,\theta) = r^2 \cos 2\theta$
   (d) $u(r,\theta) = \ln r$

   Solution:

   (a)

   $$\nabla^2(x^5 - 10x^3y^2 + 5xy^4) = (20x^3 - 60xy^2) + (-20x^3 + 60xy^2) = 0,$$

   throughout the open unit disk, so this is a solution.

   (b)

   $$\nabla^2(\cosh x \sin y + \cos x \sinh y) = (\cosh x \sin y - \cos x \sinh y) + (-\cosh x \sin y + \cos x \sinh y) = 0,$$

   throughout the open unit disk, so this is a solution.

   (c) $u(r,\theta) = r^2 \cos 2\theta = r^2\cos^2\theta - r^2 \sin\theta = x^2 - y^2$, and

   $$\nabla^2(x^2 - y^2) = 2 - 2 = 0$$

   throughout the open unit disk, so this is a solution.

   (d) Because u is not defined—let alone differentiable—throughout the entire unit disk (we may not allow $r = 0$), u is not a solution on the open unit disk.

5. Show how to use separation of variables to find the Fourier expansion of the solution for the interior Dirichlet problem on the polar unit disk $\{(r,\theta)\colon 0 \le r < 1\}$:

$$\begin{aligned} \nabla^2 u &= 0, & 0 \le r < 1 \\ u(1,\theta) &= g(\theta), \end{aligned}$$

where g is continuous and periodic of period 2π. You may assume that the general solution to the Euler equation $r^2R'' + rR' - \lambda^2 R = 0$ is $R = a + b\ln r$ when $\lambda = 0$ and $R = ar^\lambda + br^{-\lambda}$ when $\lambda \neq 0$. It may also be helpful to recall that, in polar coordinates,

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Solution: An outline of the solution is on page 263 of the text, and a more detailed discussion should be in your classroom notes.