**MTH 3440, Partial Differential Equations—Final Exam** **May 11, 2009**

**Instructions:** Work the following problems; give your reasoning and show your supporting calculations. Your paper is due at 12:50 pm.

1. Which of the following are solutions of the partial differential equation $u_t = \alpha^2 u_{xx}$? (Note: No boundary conditions; no initial condition.) *Give your reasoning.*

   (a) $u(x,t) = e^{-\lambda^2\alpha^2 t}(\cos\lambda x - \sin\lambda x)$, where $\lambda$ is an arbitrary constant.

   (b) $u(x,t) = 3x - 2$.

   (c) $u(x,t) = 2e^{\lambda^2\alpha^2 t}\tan 2\lambda x$, where $\lambda$ is an arbitrary constant.

   (d) $u(x,t) = 4\alpha^2 e^{4\alpha^2 t}\cosh 2x$.

2. Solve:

$$\begin{aligned} u_{tt} &= 4u_{xx}, & -\infty < x < \infty, 0 < t \\ u(x,0) &= 2\sin x\cos x, & -\infty < x < \infty \\ u_t(x,0) &= \cos x, & -\infty < x < \infty. \end{aligned}$$

3. Solve the Cauchy problem:

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} - u = 0;$$

$$u(x,1) = f(x),$$

   where $f$ is a continuously differentiable function.

4. Show how to transform the IBVP

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, & 0 < x < 1,\ 0 < t < \infty, \\ u(0,t) &= \cos\pi t, & 0 < t < \infty, \\ u(1,t) &= \sin\pi t, & 0 < t < \infty, \\ u(x,0) &= x, & 0 < x < 1, \end{aligned}$$

   into an IBVP with homogeneous boundary conditions. (*Do not* attempt to solve the transformed problem.)

5. Show how to use separation of variables to find the Fourier expansion of the solution for the interior Dirichlet problem on the polar unit disk $\{(r,\theta)\colon 0 \le r < 1\}$:

$$\begin{aligned} \nabla^2 u &= 0, & 0 \le r < 1 \\ u(1,\theta) &= g(\theta), \end{aligned}$$

   where $g$ is continuous and periodic of period $2\pi$. You may assume that the general solution to the Euler equation $r^2R'' + rR' - \lambda^2 R = 0$ is $R = a + b\ln r$ when $\lambda = 0$ and $R = ar^\lambda + br^{-\lambda}$ when $\lambda \neq 0$. It may also be helpful to recall that, in polar coordinates,

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.$$